Dr Oliver Mathematics
Advanced Subsidiary Paper 1: Pure Mathematics
November 2021: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. Using algebra, solve the inequality (3)

$$x^2 - x > 20,$$

writing your answer in set notation.

Solution

$$x^2 - x > 20 \Rightarrow x^2 - x - 20 > 0$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -20\end{array}\right\} -5, +4$$

$$\Rightarrow (x-5)(x+4) > 0.$$

We need a 'table of signs':

	$x < -4$	$x = -4$	$-4 < x < 5$	$x = 5$	$x > 5$
$x + 4$	$-$	0	$+$	$+$	$+$
$x - 5$	$-$	$-$	$-$	0	$+$
$(x+4)(x-5)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{\{x \in \mathbb{R} | x < -4\} \cup \{x \in \mathbb{R} | x > 5\}}}.$$

2. Given (3)

$$\frac{9^{x-1}}{3^{y+2}} = 81,$$

express y in terms of x, writing your answer in simplest form.

Solution

$$\begin{aligned}\frac{9^{x-1}}{3^{y+2}} = 81 &\Rightarrow \frac{(3^2)^{x-1}}{3^{y+2}} = 3^4 \\ &\Rightarrow \frac{3^{2(x-1}}{3^{y+2}} = 3^4 \\ &\Rightarrow \frac{3^{2x-2}}{3^{y+2}} = 3^4 \\ &\Rightarrow 3^{(2x-2)-(y+2)} = 3^4 \\ &\Rightarrow 3^{2x-y-4} = 3^4 \\ &\Rightarrow 2x - y - 4 = 4 \\ &\Rightarrow \underline{\underline{y = 2x - 8}}.\end{aligned}$$

3. Find (4)

$$\int \frac{3x^4 - 4}{2x^3}\,\mathrm{d}x,$$

writing your answer in simplest form.

Solution

$$\begin{aligned}\int \frac{3x^4 - 4}{2x^3}\,\mathrm{d}x &= \int \left(\tfrac{3}{2}x - 2x^{-3}\right)\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{3}{4}x^2 + x^{-2} + c}}.\end{aligned}$$

4. (In this question the unit vectors **i** and **j** are due east and due north respectively.)

A stone slides horizontally across ice.

Initially the stone is at the point $A(-24\mathbf{i} - 10\mathbf{j})$ m relative to a fixed point O.

After 4 seconds the stone is at the point $B(12\mathbf{i} + 5\mathbf{j})$ m relative to the fixed point O.

The motion of the stone is modelled as that of a particle moving in a straight line at constant speed.

Using the model,

(a) prove that the stone passes through O, (2)

Solution

Let $\overrightarrow{OP}$ be any point on the line joining A and B. Then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \lambda\overrightarrow{AB} \\ &= \begin{pmatrix} -24 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 12 - (-24) \\ 5 - (-10) \end{pmatrix} \\ &= \begin{pmatrix} -24 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 36 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} -24 + 36\lambda \\ -10 + 15\lambda \end{pmatrix}.\end{aligned}$$

What do we notice? At

$$\lambda = \tfrac{2}{3} \Rightarrow \overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

hence, the stone passes through O.

(b) calculate the speed of the stone. (3)

Solution

Well, let us call the velocity $\mathbf{v}$. Then

$$\begin{aligned}\mathbf{v} &= \frac{\begin{pmatrix} 12 \\ 5 \end{pmatrix} - \begin{pmatrix} -24 \\ -10 \end{pmatrix}}{4} \\ &= \frac{\begin{pmatrix} 36 \\ 15 \end{pmatrix}}{4} \\ &= \begin{pmatrix} 9 \\ 3\frac{3}{4} \end{pmatrix}\end{aligned}$$

and the speed is

$$\begin{aligned}|\mathbf{v}| &= \sqrt{9^2 + (3\tfrac{3}{4})^2} \\ &= \underline{\underline{9.75 \text{ ms}^{-1}}}.\end{aligned}$$

5. Figure 1 shows part of the curve with equation

$$y = 3x^2 - 2.$$

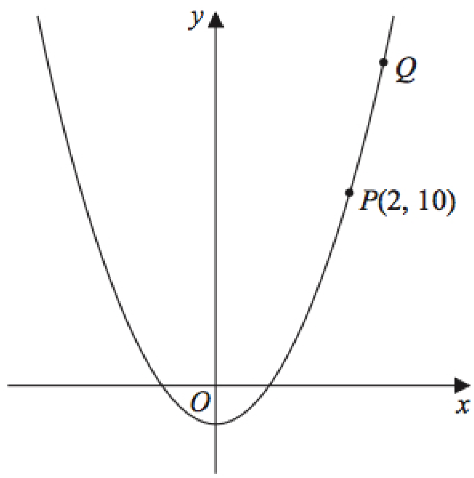

Figure 1: $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P. (2)

Solution

$$y = 3x^2 - 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6 \times 2 = \underline{\underline{12}}.$$

The point Q with x-coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ, giving your answer in terms of h in simplest form. (3)

Solution

$$\text{Gradient}_{PQ} = \frac{[3(2+h)^2 - 2] - [3(2)^2 - 2]}{(2+h) - 2}$$

$\times$	2	$+h$
2	4	$+2h$
$+h$	$+2h$	$+h^2$

$$\begin{aligned}
&= \frac{[3(4 + 4h + h^2) - 2] - 10}{h} \\
&= \frac{(10 + 12h + 3h^2) - 10}{h} \\
&= \frac{12h + 3h^2}{h} \\
&= \underline{\underline{12 + 3h}}.
\end{aligned}$$

(c) Explain briefly the relationship between part (b) and the answer to part (a). (1)

Solution
Well, let

$$f(x) = 3x^2 - 2.$$

Now,

$$\begin{aligned}
\lim_{h \to 0}(\text{gradient}_{PQ}) &= \lim_{h \to 0}\left(\frac{f(2 + h) - f(2)}{h}\right) \\
&= \lim_{h \to 0}(12 + 3h) \\
&= 12 \\
&= f'(2).
\end{aligned}$$

So, the gradient, as $h \to 0$, of PQ is precisely the same as $f'(2)$.

6. (a) Using algebra, find all solutions of the equation (3)

$$3x^3 - 17x^2 - 6x = 0.$$

Solution

$$3x^3 - 17x^2 - 6x = 0 \Rightarrow x(3x^2 - 17x - 6) = 0$$

add to: -17
multiply to: $(+3) \times (-6) = -18$ $\Big\}\; -18,\ +1$

$$\begin{aligned} &\Rightarrow x[3x^2 - 18x + x - 6] = 0 \\ &\Rightarrow x[3x(x-6) + 1(x-6)] = 0 \\ &\Rightarrow x(3x+1)(x-6) = 0 \\ &\Rightarrow \underline{\underline{x = -\tfrac{1}{3},\ x = 0,,\ \text{ or } x = 6}}. \end{aligned}$$

(b) Hence find all real solutions of (3)

$$3(y-2)^6 - 17(y-2)^4 - 6(y-2)^2 = 0.$$

Solution
Let $x = (y-2)^2$. Then

$$\begin{aligned} x = (y-2)^2 &\Rightarrow y - 2 = \pm\sqrt{x} \\ &\Rightarrow y = 2 \pm \sqrt{x}. \end{aligned}$$

The first case, $x = -\frac{1}{3}$, we cannot solve $(y-2)^2 = -\frac{1}{3}$ using real numbers. But we can do the rest!

$$\underline{\underline{y = 2 \text{ or } y = 2 \pm \sqrt{6}}}.$$

7. A parallelogram $PQRS$ has area 50 cm^2.

Given

- PQ has length 14 cm
- QR has length 7 cm, and
- angle SPQ is obtuse,

find

(a) the size of angle SPQ, in degrees, to 2 decimal places, (3)

Solution

We need the area formula:

$$\begin{aligned} & 2 \times \tfrac{1}{2} \times PQ \times QR \times \sin SPQ = 50 \\ \Rightarrow \quad & 14 \times 7 \times \sin PQR = 50 \\ \Rightarrow \quad & 98 \sin SPQ = 50 \\ \Rightarrow \quad & \sin SPQ = \tfrac{25}{49} \\ \Rightarrow \quad & \angle SPQ = 30.677\,424\,47,\ 180 - 30.677\,424\,47 \text{ (FCD)} \\ \Rightarrow \quad & \angle SPQ = 30.677\,424\,47,\ 149.322\,575\,5 \text{ (FCD)}. \end{aligned}$$

But the angle is obtuse! Hence,

$$\angle SPQ = \underline{\underline{149.32^\circ \text{ (2 dp)}}}.$$

(b) the length of the diagonal SQ, in cm, to one decimal place. (2)

Solution

We need the cosine rule:

$$\begin{aligned} & SQ^2 = QR^2 + RS^2 - 2 \times QR \times RS \times \cos QRS \\ \Rightarrow \quad & SQ^2 = 7^2 + 14^2 - 2 \times 7 \times 14 \times \cos 149.322\ldots \\ \Rightarrow \quad & SQ^2 = 413.570\,46 \text{ (FCD)} \\ \Rightarrow \quad & SQ = 20.336\,431\,84 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{SQ = 20.3 \text{ cm (1 dp)}}}. \end{aligned}$$

8.

$$\mathrm{g}(x) = (2 + ax)^8, \text{ where } a \text{ is a constant.}$$

Given that one of the terms in the binomial expansion of $\mathrm{g}(x)$ is $3\,402x^5$,

(a) find the value of a. (4)

Solution

We want to find the coefficient of x^5:

$$\begin{aligned}\binom{8}{5}(2^3)(a^5) = 3\,402 &\Rightarrow 56(8)(a^5) = 3\,402 \\ &\Rightarrow 448a^5 = 3\,402 \\ &\Rightarrow a^5 = 7.593\,75 \\ &\Rightarrow a = \sqrt[5]{7.593\,75} \\ &\Rightarrow \underline{\underline{a = 1.5}}.\end{aligned}$$

Using this value of a,

(b) find the constant term in the expansion of (3)

$$\left(1 + \frac{1}{x^4}\right)(2 + ax)^8.$$

Solution

$$\begin{aligned}\left(1 + \frac{1}{x^4}\right)(2 + ax)^8 &= (2 + ax)^8 + \frac{1}{x^4}(2 + ax)^8 \\ &= (2^8 + \ldots) + \frac{1}{x^4}\left(\ldots + \binom{8}{4}(2^4)(1.5x)^4 + \ldots\right) \\ &= (256 + \ldots) + \frac{1}{x^4}\left(\ldots + 5\,670x^4 + \ldots\right) \\ &= (256 + \ldots) + (\ldots + 5\,670 + \ldots)\,;\end{aligned}$$

hence, the constant term is

$$256 + 5\,760 = \underline{\underline{5\,926}}.$$

9. Find the value of the constant k, $0 < k < 9$, such that (4)

$$\int_k^9 \frac{6}{\sqrt{x}}\,\mathrm{d}x = 20.$$

Solution

$$\begin{aligned}\int_k^9 \frac{6}{\sqrt{x}}\,\mathrm{d}x = 20 &\Rightarrow \int_k^9 6x^{-\frac{1}{2}}\,\mathrm{d}x = 20\\ &\Rightarrow \left[12x^{\frac{1}{2}}\right]_{x=k}^{9} = 20\\ &\Rightarrow 12(9^{\frac{1}{2}}) - 12(k^{\frac{1}{2}}) = 20\\ &\Rightarrow 36 - 12(k^{\frac{1}{2}}) = 20\\ &\Rightarrow 12(k^{\frac{1}{2}}) = 16\\ &\Rightarrow k^{\frac{1}{2}} = \tfrac{4}{3}\\ &\Rightarrow k = (\tfrac{4}{3})^2\\ &\Rightarrow \underline{\underline{k = \tfrac{16}{9} \text{ or } 1\tfrac{7}{9}}}.\end{aligned}$$

10. A student is investigating the following statement about natural numbers.

"$(n^3 - n)$ is a multiple of 4."

(a) Prove, using algebra, that the statement is true for all odd numbers. (4)

Solution

"For all odd numbers"? Let $n = 2k + 1$: then

$$\begin{aligned}n^3 - n &= (2k+1)^3 - (2k+1)\\ &= (8k^3 + 12k^2 + 6k + 1) - (2k+1)\\ &= 8k^3 + 12k^2 + 4k\\ &= 4(k^3 + 3k^2 + k),\end{aligned}$$

which is a $\underline{\underline{\text{multiple of 4}}}$.

(b) Use a counterexample to show that the statement is not always true. (1)

Solution

E.g., $n = 2$:

$$2^3 - 2 = 8 - 2 = 6 = 4 \times 1\tfrac{1}{2}.$$

11. The owners of a nature reserve decided to increase the area of the reserve covered by trees.

Tree planting started on 1st January 2005.

The area of the nature reserve covered by trees, A km^2, is modelled by the equation

$$A = 80 - 45e^{ct},$$

where c is a constant and t is the number of years after 1st January 2005.

Using the model,

(a) find the area of the nature reserve that was covered by trees just before tree planting started. (1)

Solution

$$t = 0 \Rightarrow A = 80 - 45 = \underline{\underline{35 \text{ km}^2}}.$$

On 1st January 2019 an area of 60 km^2 of the nature reserve was covered by trees.

(b) Use this information to find a complete equation for the model, giving your value of c to 3 significant figures. (4)

Solution

$$\begin{aligned} t = 14, A = 60 &\Rightarrow 60 = 80 - 45e^{14c} \\ &\Rightarrow 45e^{14c} = 20 \\ &\Rightarrow e^{14c} = \tfrac{4}{9} \\ &\Rightarrow 14c = \ln \tfrac{4}{9} \\ &\Rightarrow c = \tfrac{1}{14} \ln \tfrac{4}{9} \\ &\Rightarrow c = -0.057\,923\,586\,87 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{c = -0.057\,9 \text{ (3 sf)}}}. \end{aligned}$$

On 1st January 2020, the owners of the nature reserve announced a long-term plan to have 100 km^2 of the nature reserve covered by trees.

(c) State a reason why the model is not appropriate for this plan. (1)

Solution
E.g., the maximum area covered by trees is only 80 km^2.

12. (a) (i) Solve, for $0^\circ < \theta \leqslant 450^\circ$, the equation (5)

$$5\cos^2\theta = 6\sin\theta,$$

giving your answers to one decimal place.

Solution
Use

$$\cos^2\theta \equiv 1 - \sin^2\theta :$$

$$\begin{aligned}5\cos^2\theta = 6\sin\theta &\Rightarrow 5(1-\sin^2\theta) = 6\sin\theta \\ &\Rightarrow 5 - 5\sin^2\theta = 6\sin\theta \\ &\Rightarrow 5\sin^2\theta + 6\sin\theta - 5 = 0\end{aligned}$$

Now, $b^2 - 4ac = 136$ and so we use the quadratic formula: $a = 5$, $b = 6$, and $c = -5$:

$$\begin{aligned}\sin\theta &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4\times 5\times(-5)}}{2\times 5} \\ &= \frac{-6\pm\sqrt{136}}{10}.\end{aligned}$$

$\underline{\sin\theta = \frac{-6-\sqrt{136}}{10}}$: Well, $\sin\theta < -1$ and so we have no solutions.
$\underline{\sin\theta = \frac{-6+\sqrt{136}}{10}}$:

$$\begin{aligned}\theta &= 34.484\,994\,24, 145.515\,005\,8, 394.484\,994\,24 \text{ (FCD)} \\ &= \underline{\underline{34.5^\circ, 145.1^\circ, 394.5^\circ \text{ (1 dp)}}}.\end{aligned}$$

(ii) A student's attempt to solve the question (2)

"Solve, for $-90^\circ < x < 90^\circ$, the equation $3\tan x - 5\sin x = 0$"

is set out below.

$$3\tan x - 5\sin x = 0$$

$$3\frac{\sin x}{\cos x} - 5\sin x = 0$$

$$3\sin x - 5\sin x\cos x = 0$$

$$3 - 5\cos x = 0$$

$$\cos x = \frac{3}{5}$$

$$x = 53.1°$$

Identify two errors or omissions made by this student, giving a brief explanation of each.

Solution

E.g., in the fourth line, it should have been $\sin x(3 - 5\cos x) = 0$ and hence they miss the solution $\sin x = 0 \Rightarrow x = 0$.

in the fifth line: "$-90° < x < 90°$" but $\cos x = \frac{3}{5} \Rightarrow x = \pm 53.1$

The first four positive solutions, in order of size, of the equation

$$\cos(5\alpha + 40°) = \tfrac{3}{5}$$

are α_1, α_2, α_3, and α_4.

(b) Find, to the nearest degree, the value of α_4. (2)

Solution

$$\begin{aligned} \cos(5\alpha + 40°) = \tfrac{3}{5} &\Rightarrow 5\alpha + 40° = 53.130\ldots, 306.869\ldots, 413.130\ldots, 666.869\ldots \\ &\Rightarrow 5\alpha = 13.130\ldots, 266.869\ldots, 373.130\ldots, 626.869\ldots, \ldots \\ &\Rightarrow \alpha = 2.626\ldots, 53.373\ldots, 74.626\ldots, 125.373\ldots, \ldots; \end{aligned}$$

hence,

$$\alpha_4 = \underline{\underline{125° \text{ (nearest degree)}}}.$$

13. The resting heart rate, h, of a mammal, measured in beats per minute, is modelled by the equation

$$h = pm^q,$$

where p and q are constants and m is the mass of the mammal measured in kg.

Figure 2 illustrates the linear relationship between $\log_{10} h$ and $\log_{10} m$.

The line meets the vertical $\log_{10} h$ axis at 2.25 and has a gradient of -0.235.

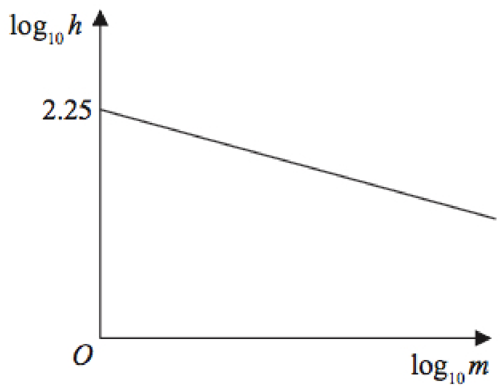

Figure 2: $\log_{10} h$ and $\log_{10} m$

(a) Find, to 3 significant figures, the value of p and the value of q. (3)

Solution

$$\begin{aligned}\log_{10} h = -0.235 \log_{10} m + 2.25 &\Rightarrow \log_{10} h + 0.235 \log_{10} m = 2.25 \\ &\Rightarrow \log_{10} h + \log_{10} m^{0.235} = 2.25 \\ &\Rightarrow \log_{10}(hm^{0.235}) = 2.25 \\ &\Rightarrow hm^{0.235} = 10^{2.25} \\ &\Rightarrow h = 10^{2.25} m^{-0.235} \\ &\Rightarrow h = 177.827\,941 m^{-0.235} \text{ (FCD);}\end{aligned}$$

hence, $p = 178$ (3 sf) and $q = -0.235$.

A particular mammal has a mass of 5 kg and a resting heart rate of 119 beats per minute.

(b) Comment on the suitability of the model for this mammal. (3)

Solution

$$\begin{aligned}h = 119, m = 5 &\Rightarrow h = 177.827\,941(5^{-0.235}) \\ &\Rightarrow h = 121.826\,579 \text{ (FCD).}\end{aligned}$$

The model predicts $h = 122$ (3 sf) and the actual value is 119. So, I would consider the model to be in good agreement with the actual mammal.

(c) With reference to the model, interpret the value of the constant p. (1)

Solution
E.g., $\log_{10} m = 0 \Rightarrow m = 1$: $p = 178$ will be the heart rate of the mammal when it is mass 1 kg.

14. A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = -3x^2 + 12x + 8.$$

(a) Write $\mathrm{f}(x)$ in the form (3)

$$a(x+b)^2 + c,$$

where a, b, and c are constants to be found.

Solution

$$\begin{aligned} \mathrm{f}(x) &= -3x^2 + 12x + 8 \\ &= 8 - 3[x^2 - 4x] \\ &= 8 - 3[(x^2 - 4x + 4) - 4] \\ &= 8 - 3[(x-2)^2 - 4] \\ &= 8 - 3(x-2)^2 + 12 \\ &= \underline{\underline{-3(x-2)^2 + 20}}; \end{aligned}$$

hence, $\underline{\underline{a = -3}}$, $\underline{\underline{b = -2}}$, and $\underline{\underline{c = 20}}$.

The curve C has a maximum turning point at M.

(b) Find the coordinates of M. (2)

Solution
$\underline{\underline{M(2, 20)}}$.

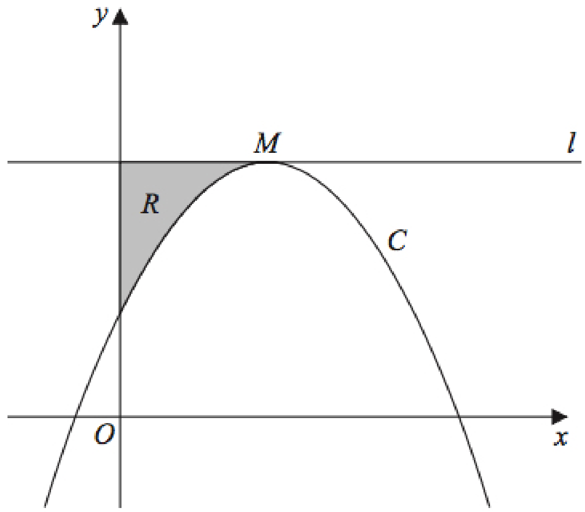

Figure 3: the curve C

Figure 3 shows a sketch of the curve C.

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 3, is bounded by C, l, and the y-axis.

(c) Using algebraic integration, find the area of R. (5)

Solution

Let us call $A(2,0)$ and $B(0,20)$. Then

$$\begin{aligned}\text{area of } R &= \text{area of } OAMB - \int_0^2 (-3x^2 + 12x + 8)\,\mathrm{d}x \\ &= (2 \times 20) - \left[-x^3 + 6x^2 + 8x\right]_{x=0}^{2} \\ &= 40 - \{(-8 + 24 + 16) - (0 + 0 + 0)\} \\ &= 40 - 32 \\ &= \underline{\underline{8}}.\end{aligned}$$

15. Figure 4 shows a sketch of a circle C with centre $N(7,4)$.

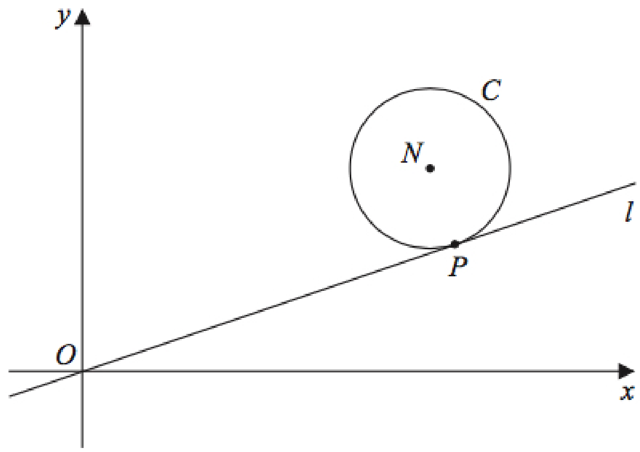

Figure 4: a sketch of a circle C

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P.

Find

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants, (2)

Solution

$\angle OPN$ is a right-angle (why?) and that makes the gradient of NP be

$$-\frac{1}{\frac{1}{3}} = -3.$$

Now, it goes through $N(7, 4)$ and so the equation is

$$\begin{aligned} y - 4 = -3(x - 7) &\Rightarrow y - 4 = -3x + 21 \\ &\Rightarrow \underline{\underline{y = -3x + 25}}; \end{aligned}$$

hence, $\underline{\underline{m = -3}}$ and $\underline{\underline{c = 25}}$.

(b) an equation for C. (4)

Solution

Simultaneous equations:

$$\begin{aligned} \tfrac{1}{3}x = -3x + 25 &\Rightarrow \tfrac{10}{3}x = 25 \\ &\Rightarrow x = 7\tfrac{1}{2} \\ &\Rightarrow y = 2\tfrac{1}{2}; \end{aligned}$$

so, $P(7\frac{1}{2}, 2\frac{1}{2})$. Well,

$$\begin{aligned} NP^2 &= (7\tfrac{1}{2} - 7)^2 + (2\tfrac{1}{2} - 4)^2 \\ &= \tfrac{1}{4} + 2\tfrac{1}{4} \\ &= 2\tfrac{1}{2} \end{aligned}$$

and the equation of C is

$$\underline{\underline{(x-7)^2 + (y-4)^2 = 2\tfrac{1}{2}}}.$$

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C.

(c) Find the value of k. (3)

Solution

Let R be the point on C where PN extended meets C. Now,

$$\begin{aligned} \overrightarrow{OR} &= \overrightarrow{OP} + \overrightarrow{PR} \\ &= \overrightarrow{OP} + 2\overrightarrow{PN} \\ &= \begin{pmatrix} 7\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + 2 \begin{pmatrix} 7 - 7\frac{1}{2} \\ 4 - 2\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 7\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} + 2 \begin{pmatrix} -\frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 6\frac{1}{2} \\ 5\frac{1}{2} \end{pmatrix}; \end{aligned}$$

hence, $R(6\frac{1}{2}, 5\frac{1}{2})$. Finally, substitute into $y = \frac{1}{3}x + k$:

$$\begin{aligned} 5\tfrac{1}{2} = \tfrac{1}{3}(6\tfrac{1}{2}) + k &\Rightarrow 5\tfrac{1}{2} = 2\tfrac{1}{6} + k \\ &\Rightarrow \underline{\underline{k = 3\tfrac{1}{3}}}. \end{aligned}$$

16. The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = ax^3 + 15x^2 - 39x + b,$$

and a and b are constants.

Given

- the point $(2, 10)$ lies on C and
- the gradient of the curve at $(2, 10)$ is -3,

(a) (i) show that the value of a is -2, (4)

Solution

$$x = 2, y = 10 \Rightarrow 10 = 8a + 60 - 78 + b$$
$$\Rightarrow 8a + b = 28 \quad (1).$$

Now,

$$y = ax^3 + 15x^2 - 39x + b \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 30x - 39$$

and

$$x = 2, \frac{\mathrm{d}y}{\mathrm{d}x} = -3 \Rightarrow -3 = 12a + 60 - 39$$
$$\Rightarrow 12a = -24$$
$$\Rightarrow \underline{\underline{a = -2}},$$

as required.

(ii) find the value of b.

Solution

From (1),

$$8(-2) + b = 28 \Rightarrow -16 + b = 28$$
$$\Rightarrow \underline{\underline{b = 44}}.$$

(b) Hence show that C has no stationary points. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -6x^2 + 30x - 39 = 0$$
$$\Rightarrow -2(3x^2 - 10x + 13) = 0.$$

Now, $a = 3$, $b = -10$, and $c = 13$:

$$b^2 - 4ac = 10^2 - 4 \times 3 \times 13 = -56 < 0;$$

hence, C has no stationary points.

(c) Write $\mathrm{f}(x)$ in the form (2)

$$(x-4)Q(x),$$

where $Q(x)$ is a quadratic expression to be found.

Solution
We use synthetic division:

4	-2	15	-39	44
	$\downarrow$	-8	28	-44
	-2	7	-11	0

There is no remainder! Hence,

$$Q(x) = \underline{\underline{-2x^2 + 7x - 11}}.$$

(d) Hence deduce the coordinates of the points of intersection of the curve with equation (2)

$$y = \mathrm{f}(0.2x)$$

and the coordinate axes.

Solution
Well,

$$y = \mathrm{f}(x) \to y = \mathrm{f}(0.2x)$$

is a stretch, scale factor 5, in the x-direction.
Hence, the coordinates of the points of intersection are $\underline{\underline{(0,44)}}$ and $5 \times 4 = 20$: $\underline{\underline{(20,0)}}$.