Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2009 November Paper 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. A function f is defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{x-1},$$

where $x > 0$

(a) State the range of f. (1)

Solution
$\mathrm{f}(x) > \mathrm{e}^{-1}$.

(b) Find an expression for f^{-1}. (2)

Solution

$$\begin{aligned} y = \mathrm{e}^{x-1} &\Rightarrow x - 1 = \ln y \\ &\Rightarrow x = 1 + \ln y \end{aligned}$$

so

$$\mathrm{f}^{-1}(x) = 1 + \ln x.$$

(c) State the domain of f^{-1}. (1)

Solution
$\mathrm{f}^{-1}(x) > \mathrm{e}^{-1}$.

2. (a) Find the first four terms, in ascending powers of x, in the expansion of (4)

$$\left(2 - \tfrac{1}{2}x\right)^6.$$

Solution

$$\left(2-\tfrac{1}{2}x\right)^6 = 2^6 + \binom{6}{1}(2)^5(-\tfrac{1}{2}x) + \binom{6}{2}(2)^4(-\tfrac{1}{2}x)^2 + \binom{6}{3}(2)^3(-\tfrac{1}{2}x)^3 + \ldots$$
$$= \underline{\underline{64 - 96x + 60x^2 - 20x^3 + \ldots}}.$$

(b) Find the coefficient of x^3 in the expansion of (2)

$$(1+x)^2\left(2-\tfrac{1}{2}x\right)^6.$$

Solution

Well,

$\times$	1	$+x$
1	1	$+x$
$+x$	$+x$	$+x^2$

so

$$(1+x)^2\left(2-\tfrac{1}{2}x\right)^6 = (1+2x+x^2)(64-96x+60x^2-20x^3+\ldots).$$

Now,

$\times$	$-96x$	$+60x^2$	$-20x^3$
1	$\ldots$	$\ldots$	$-20x^3$
$+2x$	$\ldots$	$+120x^3$	$\ldots$
$+x^2$	$-96x^3$	$\ldots$	$\ldots$

So, the coefficient of x^3 is

$$-20+120-96 = \underline{\underline{4}}.$$

3. The table shows experimental values of the variables x and y which are related by the equation

$$y = \frac{a}{x^2} + \frac{b}{x},$$

where a and b are constants.

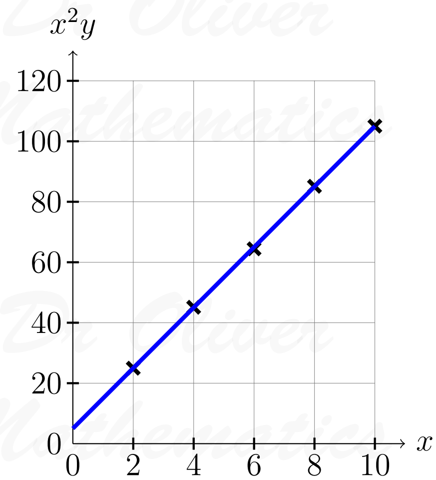
x	2	4	6	8	10
y	6.24	2.82	1.79	1.33	1.05

(a) Using graph paper, plot x^2y against x and draw a straight line graph. (3)

Solution

x	2	4	6	8	10
y	6.24	2.82	1.79	1.33	1.05
x^2y	24.96	45.12	64.44	85.12	105

and we plot the graph:



(b) Use your graph to estimate the value of a and of b. (4)

Solution

The graph goes through $(0, 5)$ and $(10, 105)$:

$$m = \frac{105 - 5}{10 - 0} = \frac{100}{10} = 10$$

and the equation is

$$x^2y - 5 = 10(x - 0) \Rightarrow x^2y = 10x + 5$$
$$\Rightarrow \underline{\underline{y = \frac{5}{x^2} + \frac{10}{x}}};$$

hence, $\underline{\underline{a = 5}}$ and $\underline{\underline{b = 10}}$.

4. Find the coordinates and the nature of the stationary points of the curve (7)

$$y = x^3 + 3x^2 - 45x + 60.$$

Solution

Well,

$$y = x^3 + 3x^2 - 45x + 60 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 6x - 45$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^2 + 6x - 45 = 0$$
$$\Rightarrow 3(x^2 + 2x - 15) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & +2 \\ \text{multiply to:} & -15\end{array}\right\} + 5,\ -3$$

$$\Rightarrow 3(x + 5)(x - 3) = 0$$
$$\Rightarrow x = -5 \text{ or } x = 3$$
$$\Rightarrow y = 235 \text{ or } y = -21.$$

Now,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 6$$

and

$$x = -5 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -24$$
$$x = 3 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 24.$$

Hence, $\underline{\underline{(-5, 235)}}$ is a $\underline{\underline{\text{maximum}}}$ and $\underline{\underline{(3, -21)}}$ is a $\underline{\underline{\text{minimum}}}$.

5. Relative to an origin O, the position vectors of points A and B are

$$\begin{pmatrix} 7 \\ 24 \end{pmatrix} \text{ and } \begin{pmatrix} 10 \\ 20 \end{pmatrix},$$

respectively.

Find

(a) the length of $\overrightarrow{OA}$, (2)

Solution

$$\begin{aligned} OA &= \sqrt{7^2 + 24^2} \\ &= \sqrt{49 + 576} \\ &= \sqrt{625} \\ &= \underline{\underline{25}}. \end{aligned}$$

(b) the length of $\overrightarrow{AB}$. (2)

Solution

Well,

$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 10 \\ 20 \end{pmatrix} - \begin{pmatrix} 7 \\ 24 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -4 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} AB &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= \underline{\underline{5}}. \end{aligned}$$

Given that ABC is a straight line and that the length of $\overrightarrow{AC}$ is equal to the length of $\overrightarrow{OA}$, find

(c) the position vector of the point C. (3)

Solution

Well,

$$\overrightarrow{AC} = k \begin{pmatrix} 3 \\ -4 \end{pmatrix},$$

for some constant k. But

$$5k = 25 \Rightarrow k = 5$$

so

$$\overrightarrow{AC} = 5\overrightarrow{AB}.$$

Now,

$$\begin{aligned} \overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + 5\overrightarrow{AB} \\ &= \begin{pmatrix} 7 \\ 24 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 22 \\ 4 \end{pmatrix}; \end{aligned}$$

hence, $\underline{\underline{C(22, 4)}}$.

6. (a) Given that (4)

$$y = x\sqrt{4x + 12},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k(x + 2)}{\sqrt{4x + 12}},$$

where k is a constant to be found.

Solution

Product rule:

$$\begin{aligned} u = x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ v = (4x + 12)^{\frac{1}{2}} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2(4x + 12)^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x)(2(4x+12)^{-\frac{1}{2}}) + (1)((4x+12)^{\frac{1}{2}})\\ &= 2x(4x+12)^{-\frac{1}{2}} + (4x+12)^{\frac{1}{2}}\\ &= (4x+12)^{-\frac{1}{2}}[2x + (4x+12)]\\ &= (4x+12)^{-\frac{1}{2}}(6x+12)\\ &= \underline{\underline{\frac{6(x+2)}{\sqrt{4x+12}}}};\end{aligned}$$

hence, $\underline{\underline{k=6}}$.

(b) Hence evaluate (3)

$$\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\,\mathrm{d}x.$$

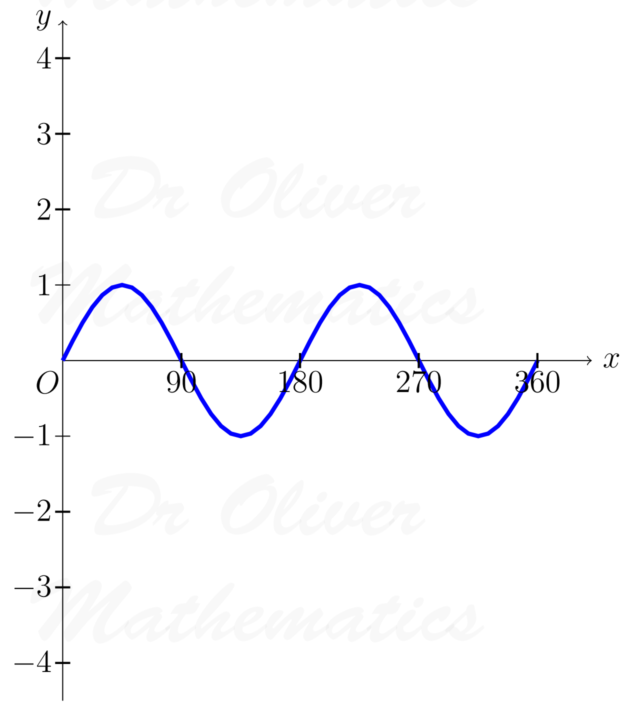
Solution
Now,

$$\begin{aligned}\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\,\mathrm{d}x &= \int_{-2}^{6} \frac{3(x+2)}{\sqrt{4x+12}}\,\mathrm{d}x\\ &= \tfrac{1}{2}\int_{-2}^{6} \frac{6(x+2)}{\sqrt{4x+12}}\,\mathrm{d}x\\ &= \tfrac{1}{2}\left[x\sqrt{4x+12}\right]_{x=-2}^{6}\\ &= \tfrac{1}{2}[36-(-4)]\\ &= \tfrac{1}{2}\times 40\\ &= \underline{\underline{20}}.\end{aligned}$$

7. (a) Using graph paper, draw the curve (3)

$$y = \sin 2x,\ 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution



In order to solve the equation

$$1 + \sin 2x = 2\cos x$$

another curve must be added to your diagram.

(b) Write down the equation of this curve and add this curve to your diagram. (3)

Solution

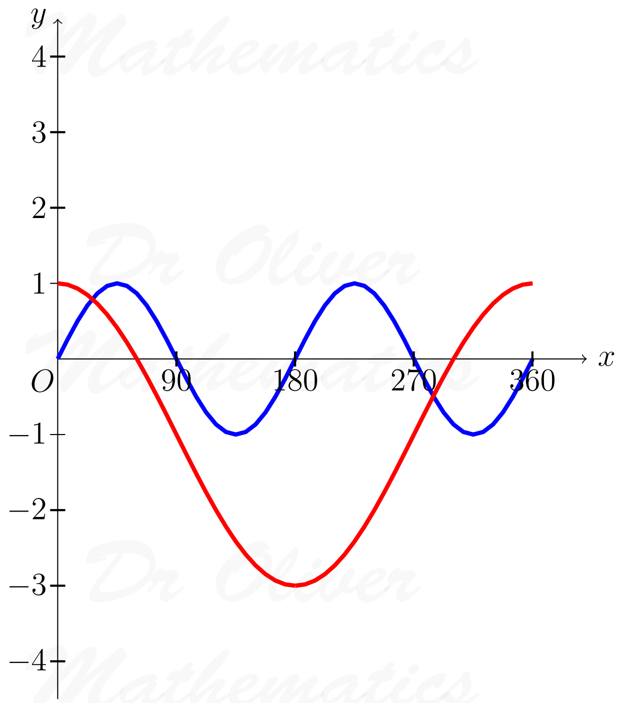
Well,

$$1 + \sin 2x = 2\cos x \Rightarrow \sin 2x = 2\cos x - 1$$

so add

$$\underline{\underline{y = 2\cos x - 1}}$$

to the graph.



(c) State the number of values of x which satisfy the equation (1)

$$1 + \sin 2x = 2\cos x,\ 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution

$\underline{\underline{2}}$.

8. It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix},\ \text{and } \mathbf{C} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

Find

(a) **AB**, (2)

Solution

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 0 & -6 \\ 10 & -12 \end{pmatrix}}}.\end{aligned}$$

(b) **BC**, (2)

Solution

$$\begin{aligned}\mathbf{BC} &= \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 11 \\ 10 \end{pmatrix}}}.\end{aligned}$$

(c) $\mathbf{A}^{-1}$, and hence find the matrix **X** such that $\mathbf{AX} = \mathbf{B}$. (4)

Solution

Well,

$$\det \mathbf{A} = 6 - (-4) = 10$$

and

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}}}.$$

Finally,

$$\begin{aligned}\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \Rightarrow \mathbf{X} &= \tfrac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \\ \Rightarrow \mathbf{X} &= \tfrac{1}{10}\begin{pmatrix} 5 & -9 \\ 0 & 12 \end{pmatrix} \\ \Rightarrow \mathbf{X} &= \underline{\underline{\begin{pmatrix} 0.5 & -0.9 \\ 0 & 1.2 \end{pmatrix}}}.\end{aligned}$$

9. A particle moves in a straight line so that, t seconds after passing through a fixed point O,

its velocity, v ms^{-1}, is given by

$$v = \frac{20}{(2t+4)^2}.$$

Find

(a) the velocity of the particle at O, (1)

Solution

$$t = 0 \Rightarrow \underline{\underline{v = 1\tfrac{1}{4} \text{ ms}^{-1}}}.$$

(b) the acceleration of the particle when $t = 3$, (3)

Solution

Well,

$$v = \frac{20}{(2t+4)^2} \Rightarrow v = 20(2t+4)^{-2}$$
$$\Rightarrow a = -80(2t+4)^{-3}$$

and

$$t = 3 \Rightarrow \underline{\underline{a = -\tfrac{2}{25} \text{ ms}^{-2}}}.$$

(c) the distance travelled by the particle in the first 8 seconds. (4)

Solution

Now,

$$v = 20(2t+4)^{-2} \Rightarrow s = -10(2t+4)^{-1} + c$$

and

$$\begin{aligned} s(8) - s(0) &= [-10(2 \times 8 + 4)^{-1} + c] - [2\tfrac{1}{2} + c] \\ &= -2; \end{aligned}$$

hence, distance travelled by the particle in the first 8 seconds is $\underline{\underline{2 \text{ m}}}$.

10. (a) Solve (4)

$$\log_{10}(7x-3) + 2\log_{10} 5 = 2 + \log_{10}(x+3).$$

Solution

$$\begin{aligned}
& \log_{10}(7x-3) + 2\log_{10} 5 = 2 + \log_{10}(x+3) \\
\Rightarrow\quad & \log_{10}(7x-3) + \log_{10} 5^2 - \log_{10}(x+3) = 2 \\
\Rightarrow\quad & \log_{10}\left(\frac{25(7x-3)}{x+3}\right) = 2 \\
\Rightarrow\quad & \frac{25(7x-3)}{x+3} = 10^2 \\
\Rightarrow\quad & 25(7x-3) = 100(x+3) \\
\Rightarrow\quad & 7x-3 = 4(x+3) \\
\Rightarrow\quad & 7x-3 = 4x+12 \\
\Rightarrow\quad & 3x = 15 \\
\Rightarrow\quad & \underline{\underline{x = 5}}.
\end{aligned}$$

(b) Use the substitution (5)

$$u = 3^x$$

to solve the equation

$$3^{x+1} + 3^{2-x} = 28.$$

Solution

$$\begin{aligned}
3^{x+1} + 3^{2-x} = 28 &\Rightarrow 3^x \times 3^1 + 3^2 \div 3^x = 28 \\
&\Rightarrow 3u + \frac{9}{u} = 28 \\
&\Rightarrow 3u^2 + 9 = 28u \\
&\Rightarrow 3u^2 - 28u + 9 = 0
\end{aligned}$$

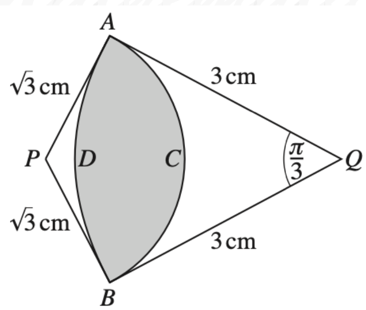
$$\left.\begin{array}{ll}\text{add to:} & -28 \\ \text{multiply to:} & (+3) \times (+9) = +27\end{array}\right\} -27,\ -1$$

e.g.,

$$\begin{aligned} &\Rightarrow 3u^2 - 27u - u + 9 = 0 \\ &\Rightarrow 3u(u-9) - 1(u-9) = 0 \\ &\Rightarrow (3u-1)(u-9) = 0 \\ &\Rightarrow u = \tfrac{1}{3} \text{ or } u = 9 \\ &\Rightarrow 3^x = \tfrac{1}{3} \text{ or } 3^x = 9 \\ &\Rightarrow \underline{\underline{x = -1 \text{ or } x = 2}}. \end{aligned}$$

EITHER

11. In the diagram, ACB is an arc of a circle with centre P, and ADB is an arc of a circle with centre Q.



- Angle $AQB = \frac{1}{3}\pi$.
- $AQ = BQ = 3$ cm.
- $AP = BP = \sqrt{3}$ cm.

(a) Show that angle $APB = \frac{2}{3}\pi$. (2)

Solution

Now,

$$\begin{aligned} & AB^2 = AQ^2 + BQ^2 - 2 \times AQ \times BQ \times \cos AQB \\ \Rightarrow\quad & AB^2 = 3^2 + 3^2 - 2 \times 3 \times 3 \times \cos \tfrac{1}{3}\pi \\ \Rightarrow\quad & AB^2 = 9 \\ \Rightarrow\quad & AB = 3. \end{aligned}$$

Next,

$$\begin{aligned} & AB^2 = AP^2 + BP^2 - 2 \times AP \times BP \times \cos APB \\ \Rightarrow\ & 3^2 = (\sqrt{3})^2 + (\sqrt{3})^2 - 2 \times \sqrt{3} \times \sqrt{3} \times \cos APB \\ \Rightarrow\ & 9 = 6 - 6 \cos APB \\ \Rightarrow\ & 6 \cos APB = -3 \\ \Rightarrow\ & \cos APB = -\tfrac{1}{2} \\ \Rightarrow\ & \underline{\underline{\angle APB = \tfrac{2}{3}\pi}}, \end{aligned}$$

as required.

(b) Find the perimeter of the shaded region. (3)

Solution
Well,

$$\begin{aligned} \text{perimeter} &= \text{perimeter of arc } ACB + \text{perimeter of arc } ADC \\ &= (\tfrac{2}{3}\pi \times \sqrt{3}) + (\tfrac{1}{3}\pi \times 3) \\ &= \tfrac{2}{3}\sqrt{3}\pi + \pi \\ &= \tfrac{2}{3}\sqrt{3}\pi + \pi \\ &= \underline{\underline{(\tfrac{2}{3}\sqrt{3} + 1)\pi}}. \end{aligned}$$

(c) Find the area of the shaded region. (5)

Solution
First,

$$\begin{aligned} \text{area of the shaded region } ABDA &= (\tfrac{1}{2} \times 3^2 \times \tfrac{1}{3}\pi) - (\tfrac{1}{2} \times 3^2 \times \sin \tfrac{1}{3}\pi) \\ &= \tfrac{3}{2}\pi - \tfrac{9}{4}\sqrt{3}. \end{aligned}$$

Second,

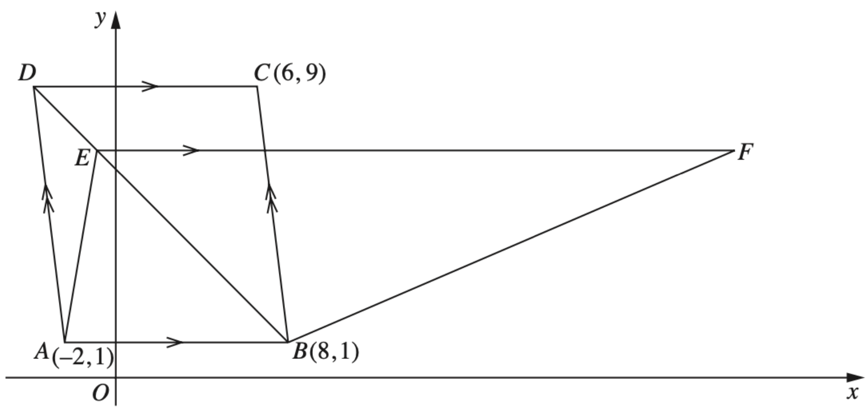
$$\begin{aligned} \text{area of the shaded region } ABCA &= (\tfrac{1}{2} \times (\sqrt{3})^2 \times \tfrac{2}{3}\pi) - (\tfrac{1}{2} \times (\sqrt{3})^2 \times \sin \tfrac{2}{3}\pi) \\ &= \pi - \tfrac{3}{4}\sqrt{3}. \end{aligned}$$

Finally,

$$\begin{aligned}\text{shaded region} &= (\tfrac{3}{2}\pi - \tfrac{9}{4}\sqrt{3}) + (\pi - \tfrac{3}{4}\sqrt{3}) \\ &= \underline{\underline{\tfrac{5}{2}\pi - 3\sqrt{3}}}.\end{aligned}$$

OR

12. The diagram shows a parallelogram with vertices $A(-2, 1)$, $B(8, 1)$, $C(6, 9)$, and D.



(a) Find the coordinates of D. (2)

Solution

Well,

$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} \\ &= \overrightarrow{OA} + \overrightarrow{BC} \\ &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 6-8 \\ 9-1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 9 \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{D(-4, 9)}}$.

The point E lies on the diagonal DB such that

$$DE = \tfrac{1}{4}DB.$$

(b) Find the coordinates of E. (2)

Solution

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{OB} + \overrightarrow{BE} \\ &= \overrightarrow{OB} + \tfrac{3}{4}\overrightarrow{BD} \\ &= \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \tfrac{3}{4}\begin{pmatrix} -4-8 \\ 9-1 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \tfrac{3}{4}\begin{pmatrix} -12 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 7 \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{E(-1, 7)}}$.

- The point F is such that EF is parallel to AB.
- The area of trapezium $AEFB$ is

$$\tfrac{3}{2} \times (\text{the area of parallelogram } ABCD).$$

(c) Find the coordinates of F. (6)

Solution

Well,

$$\begin{aligned}\text{area of trapezium } ABCD &= [8-(-2)] \times (9-1) \\ &= 10 \times 8 \\ &= 80\end{aligned}$$

and

$$\begin{aligned}\text{area of trapezium } AEFB &= \tfrac{3}{2} \times 80 \\ &= 120.\end{aligned}$$

Next,

$$\begin{aligned} A = \tfrac{1}{2}(a+b)h &\Rightarrow 120 = \tfrac{1}{2}(10+EF)(7-1) \\ &\Rightarrow 10 + EF = 40 \\ &\Rightarrow EF = 30. \end{aligned}$$

Finally,

$$-1 + 30 = 29$$

so $\underline{\underline{F(29, 7)}}$.