

Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2012 November Paper 1 Variant 2: Calculator
2 hours

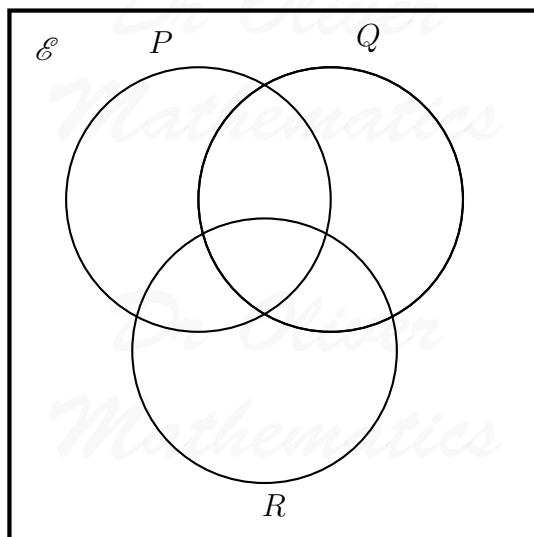
The total number of marks available is 80.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

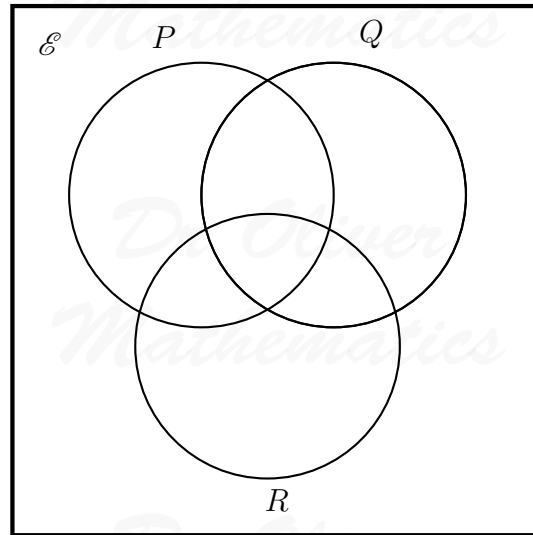
You must write down all the stages in your working.

1. (a) On the Venn diagrams below, shade the region corresponding to the set given below each Venn diagram. (2)

$$P \cup (Q \cap R)$$

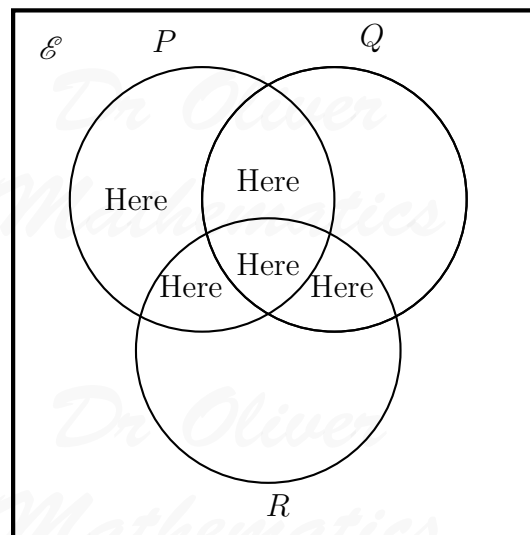


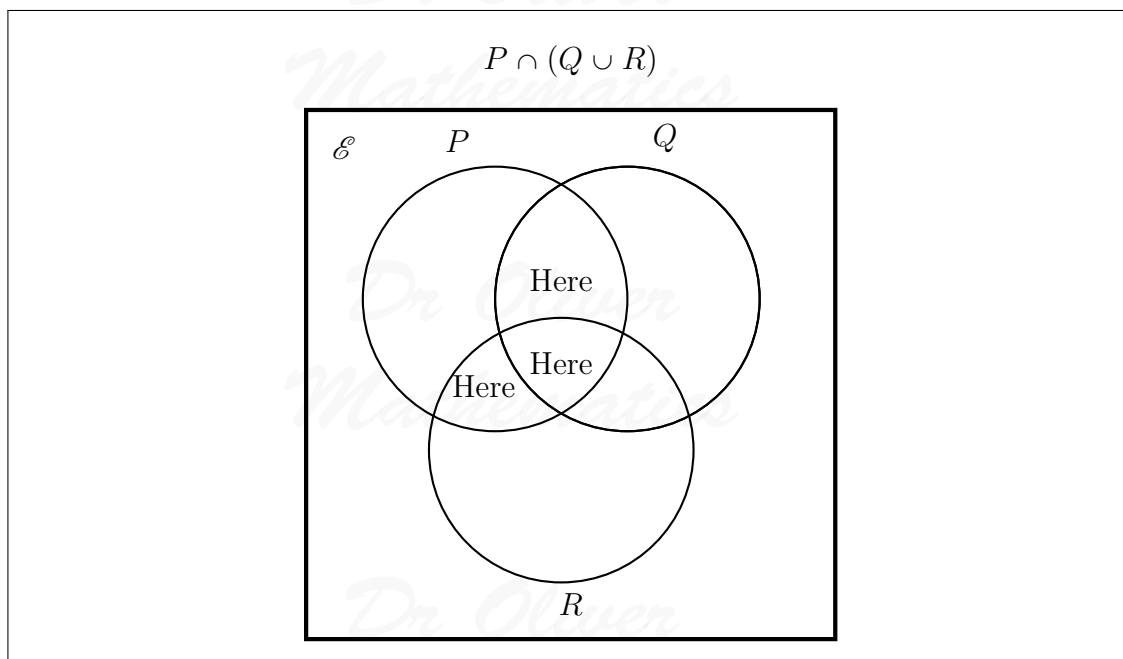
$$P \cap (Q \cup R)$$



Solution

$$P \cup (Q \cap R)$$





(b) It is given that sets $\mathcal{E}$, B , S , and F are such that

- $\mathcal{E} = \{\text{students in a school}\}$,
- $B = \{\text{students who are boys}\}$,
- $S = \{\text{students in the swimming team}\}$, and
- $F = \{\text{students in the football team}\}$.

Express each of the following statements in set notation.

(i) All students in the football team are boys.

(1)

Solution

$F \subset B$.

(ii) There are no students who are in both the swimming team and the football team.

(1)

Solution

$S \cap F = \emptyset$.

2. The rate of change of a variable x with respect to time t is

$$4 \cos^2 t.$$

(a) Find the rate of change of x with respect to t when $t = \frac{1}{6}\pi$.

(1)

Solution

Well,

$$t = \frac{1}{6}\pi \Rightarrow \underline{\underline{\frac{dx}{dt} = 3.}}$$

The rate of change of a variable y with respect to time t is

$$3 \sin t.$$

- (b) Using your result from part (a), find the rate of change of y with respect to x when $t = \frac{1}{6}\pi$. (3)

Solution

Now,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{3 \sin t}{3} \\ &= \sin t. \end{aligned}$$

Finally,

$$t = \frac{1}{6}\pi \Rightarrow \underline{\underline{\frac{dy}{dx} = \frac{1}{2}.}}$$

3. A committee of 7 members is to be selected from 6 women and 9 men.

Find the number of different committees that may be selected if

- (a) there are no restrictions, (1)

Solution

Well,

$$6 + 9 = 15$$

and

$$\binom{15}{7} = \underline{\underline{6435.}}$$

- (b) the committee must consist of 2 women and 5 men, (2)

Solution

$$\binom{9}{5} \times \binom{6}{2} = \underline{\underline{1\,890}}.$$

- (c) the committee must contain at least 1 woman. (3)

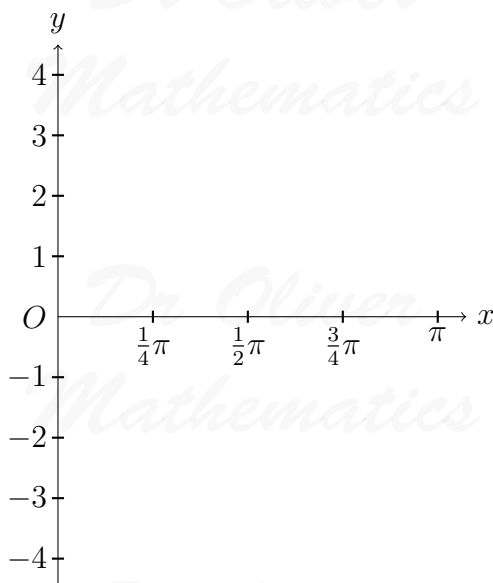
Solution

Now,

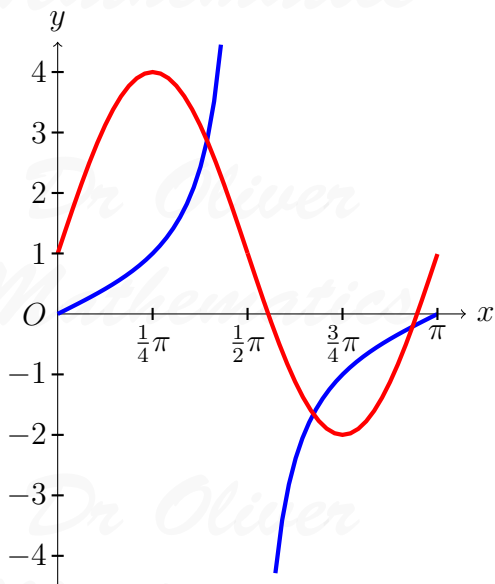
$$\begin{aligned} \text{contain at least 1 woman} &= 6\,435 - \text{all men} \\ &= 6\,435 - \binom{9}{7} \\ &= 6\,435 - 36 \\ &= \underline{\underline{6\,399}}. \end{aligned}$$

4. (a) On the axes below sketch, for $0 \leq x \leq \pi$, the graphs of (3)

$$y = \tan x \text{ and } y = 1 + 3 \sin 2x.$$



Solution



Write down

(b) the coordinates of the stationary points on the curve

(2)

$$y = 1 + 3 \sin 2x \text{ for } 0 \leq x \leq \pi,$$

Solution

Well,

$$y = 1 + 3 \sin 2x \Rightarrow \frac{dy}{dx} = 6 \cos 2x$$

and

$$\begin{aligned} \frac{dy}{dx} = 0 &\Rightarrow 6 \cos 2x = 0 \\ &\Rightarrow \cos 2x = 0 \end{aligned}$$

$$0 \leq x \leq \pi \Rightarrow 0 \leq 2x \leq 2\pi:$$

$$\begin{aligned} &\Rightarrow 2x = \frac{1}{4}\pi \text{ or } 2x = \frac{3}{4}\pi \\ &\Rightarrow x = \frac{1}{8}\pi \text{ or } x = \frac{3}{8}\pi \\ &\Rightarrow y = 4 \text{ or } y = -2; \end{aligned}$$

hence, the coordinates of the stationary points are $(\frac{1}{8}\pi, 4)$ and $(\frac{3}{8}\pi, -2)$.

- (c) the number of solutions of the equation (1)

$$\tan x = 1 + 3 \sin 2x \text{ for } 0 \leq x \leq \pi.$$

Solution

3.

5. A pilot flies his plane directly from a point A to a point B , a distance of 450 km.

The bearing of B from A is 030° .

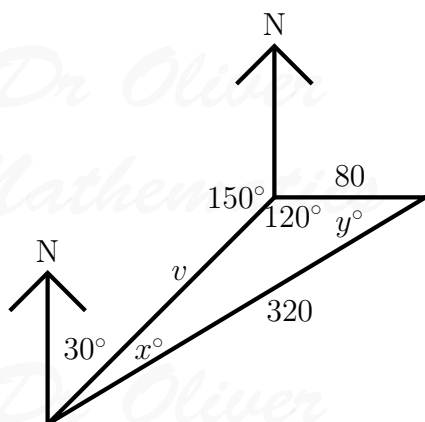
A wind of 80 kmh^{-1} is blowing from the east.

Given that the plane can travel at 320 kmh^{-1} in still air, find

- (a) the bearing on which the plane must be steered, (4)

Solution

We need a picture:



Sine rule:

$$\frac{\sin x^\circ}{80} = \frac{\sin 120^\circ}{320} \Rightarrow \sin x^\circ = \frac{80 \sin 120^\circ}{320}$$

$$\Rightarrow x = 12.50391662 \text{ (FCD)}.$$

Hence,

$$\begin{aligned}\text{bearing} &= 030 + 12.503\dots \\ &= 042.503\,916\,62 \text{ (FCD)} \\ &= \underline{\underline{043^\circ \text{ (3 sf)}}}.\end{aligned}$$

(b) the time taken to fly from A to B .

(4)

Solution

Let the speed be $v \text{ kmh}^{-1}$. Sine rule:

$$\begin{aligned}\frac{v}{\sin(180 - 120 - 12.503\dots)} &= \frac{320}{\sin 120^\circ} \Rightarrow v = \frac{320 \sin 47.496\dots}{\sin 120^\circ} \\ &\Rightarrow v = 272.409\,987 \text{ (FCD)}.\end{aligned}$$

Hence,

$$\begin{aligned}\text{time taken} &= \frac{450}{272.409\dots} \\ &= 1.651\,921\,814 \text{ (FCD)} \\ &= \underline{\underline{1.65 \text{ hours (3 sf)}}}.\end{aligned}$$

6. In the expansion of

$$(p + x)^6,$$

where p is a positive integer, the coefficient of x^2 is equal to 1.5 times the coefficient of x^3 .

(a) Find the value of p .

(4)

Solution

Well,

$$\begin{aligned}(p + x)^6 &= \dots + \binom{6}{1}(x)^1(p)^5 + \binom{6}{2}(x)^2(p)^4 + \binom{6}{3}(x)^3(p)^3 + \dots \\ &= p^6 + 6p^5x + 15p^4x^2 + 20p^3x^3 + \dots\end{aligned}$$

So, the coefficient of x^2 is equal to 1.5 times the coefficient of x^3 which leads to

$$\begin{aligned} 15p^4 &= 1.5 \times 20p^3 \Rightarrow 15p^4 = 30p^3 \\ &\Rightarrow 15p^4 - 30p^3 = 0 \\ &\Rightarrow 15p^3(p - 2) = 0 \\ &\Rightarrow p = 0 \text{ or } p = 2. \end{aligned}$$

As $p > 0$, $p = 2$.

(b) Use your value of p to find the term independent of x in the expansion of

(3)

$$(p + x)^6 \left(1 - \frac{1}{x}\right)^2.$$

Solution

Well,

$$\left(1 - \frac{1}{x}\right)^2 = 1 - \frac{2}{x} + \frac{1}{x^2}.$$

Now,

$$(2 + x)^6 \left(1 - \frac{1}{x}\right)^2 = (64 + 192x + 240x^2 + \dots) \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)$$

$\times$	64	$+192x$	$+240x^2$
1	64	$\dots$	$\dots$
$-\frac{2}{x}$	$\dots$	-384	$\dots$
$+\frac{1}{x^2}$	$\dots$	$\dots$	$+240$

$$= \dots + (64 - 384 + 240) + \dots$$

Hence, the term independent of x is -80 .

7. A particle P moves along the x -axis such that its distance, x m, from the origin O at time t s is given by

$$x = \frac{t}{t^2 + 1}, \text{ for } t > 0.$$

- (a) Find the greatest distance of P from O .

(4)

Solution

Quotient rule:

$$u = t \Rightarrow \frac{du}{dt} = 1$$

$$v = t^2 + 1 \Rightarrow \frac{dv}{dt} = 2t$$

and

$$\begin{aligned}\frac{dx}{dt} &= \frac{(t^2 + 1)(1) - (t)(2t)}{(t^2 + 1)^2} \\ &= \frac{t^2 + 1 - 2t^2}{(t^2 + 1)^2} \\ &= \frac{1 - t^2}{(t^2 + 1)^2}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{dx}{dt} = 0 &\Rightarrow 1 - t^2 = 0 \\ &\Rightarrow (1 - t)(1 + t) = 0 \\ &\Rightarrow t = \pm 1;\end{aligned}$$

as $t > 0$, $t = 1$. Hence, the greatest distance of P from O is

$$\frac{1}{2^2 + 1} = \underline{\underline{0.5 \text{ m}}}.$$

- (b) Find the acceleration of P at the instant when P is at its greatest distance from O .

(3)

Solution

Quotient rule:

$$u = 1 - t^2 \Rightarrow \frac{d^2u}{dt^2} = -2t$$

$$v = (t^2 + 1)^2 \Rightarrow \frac{d^2v}{dt^2} = 2(t^2 + 1)$$

and

$$\frac{d^2x}{dt^2} = \frac{[(t^2 + 1)^2](-2t) - [2(t^2 + 1)][1 - t^2]}{(t^2 + 1)^4}.$$

Now,

$$t = 1 \Rightarrow \frac{d^2x}{dt^2} = -\frac{1}{2};$$

the acceleration of P is -0.5 ms^{-2} .

8. (a) Given that

$$3x^3 + 5x^2 + px + 8 \equiv (x - 2)(ax^2 + bx + c),$$

(5)

find the value of each of the integers a , b , c , and p .

Solution

We use synthetic division:

$$\begin{array}{r|rrrr} 2 & 3 & 5 & p & 8 \\ & \downarrow & 6 & 22 & 44 + 2p \\ \hline & 3 & 11 & 22 + p & 52 + 2p \end{array}$$

So since the remainder is zero,

$$\begin{aligned} 52 + 2p &= 0 \Rightarrow 2p = -52 \\ &\Rightarrow \underline{\underline{p = -26}}. \end{aligned}$$

Now,

$$3x^3 + 5x^2 - 4x + 8 = (x - 2)(3x^2 + 11x - 4);$$

hence, $a = 3$, $b = 11$, and $c = -4$.

- (b) Using the values found in part (a), factorise completely

(2)

$$3x^3 + 5x^2 + px + 8.$$

Solution

$$\left. \begin{array}{l} \text{add to:} \quad \quad \quad +11 \\ \text{multiply to:} \quad (+3) \times (-4) = -12 \end{array} \right\} + 12, -1$$

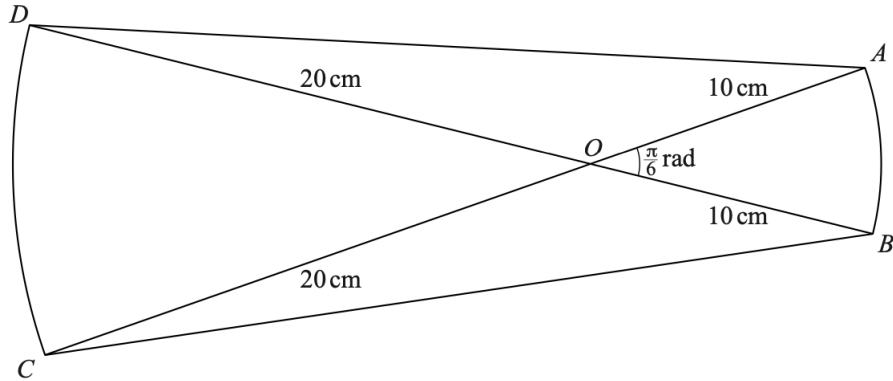
E.g.,

$$\begin{aligned} 3x^2 + 11x - 4 &= 3x^2 + 12x - x - 4 \\ &= 3x(x + 4) - 1(x + 4) \\ &= (3x - 1)(x + 4) \end{aligned}$$

and

$$3x^3 + 5x^2 - 4x + 8 = \underline{\underline{(x - 2)(3x - 1)(x + 4)}}.$$

9. The diagram shows four straight lines: AD , BC , AC , and BD .



- Lines AC and BD intersect at O such that angle AOB is $\frac{1}{6}\pi$ radians.
 - AB is an arc of the circle, centre O and radius 10 cm, and CD is an arc of the circle, centre O and radius 20 cm.
- (a) Find the perimeter of $ABCD$. (4)

Solution

Well,

$$\begin{aligned} AD^2 &= OD^2 + OA^2 - 2 \times OD \times OA \times \cos(\pi - \frac{1}{6}\pi) \\ \Rightarrow AD^2 &= 20^2 + 10^2 - 2 \times 20 \times 10 \times \cos(\frac{5}{6}\pi) \\ \Rightarrow AD^2 &= 846.410.161\ 5 \text{ (FCD)} \\ \Rightarrow AD &= 29.093\ 129\ 11 \text{ (FCD)}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{perimeter} &= \text{arc } CD + 2AD + \text{arc } AB \\ &= (\frac{1}{6}\pi \times 20) + (2 \times 29.093\dots) + (\frac{1}{6}\pi \times 10) \\ &= 73.894\ 221\ 49 \text{ (FCD)} \\ &= \underline{\underline{73.9 \text{ cm (3 sf)}}}. \end{aligned}$$

- (b) Find the area of $ABCD$. (4)

Solution

$$\begin{aligned}
\text{Area} &= \text{sector } COD + 2 \times \text{triangle } OAD + \text{sector } OAB \\
&= \left(\frac{1}{2} \times 20^2 \times \frac{1}{6}\pi\right) + 2\left[\frac{1}{2} \times 20 \times 10 \times \sin\left(\pi - \frac{1}{6}\pi\right)\right] + \left(\frac{1}{2} \times 10^2 \times \frac{1}{6}\pi\right) \\
&= 230.899\,693\,9 \text{ (FCD)} \\
&= \underline{\underline{231 \text{ cm}^2 \text{ (3 sf)}}}.
\end{aligned}$$

10. (a) Solve

$$\tan^2 x - 2 \sec x + 1 = 0 \text{ for } 0^\circ \leq x \leq 360^\circ.$$

(4)

Solution

$$\begin{aligned}
\tan^2 x - 2 \sec x + 1 = 0 &\Rightarrow (\sec^2 x - 1) - 2 \sec x + 1 = 0 \\
&\Rightarrow \sec^2 x - 2 \sec x = 0 \\
&\Rightarrow \sec x(\sec x - 2) = 0 \\
&\Rightarrow \sec x = 0 \text{ or } \sec x = 2 \\
&\Rightarrow \cos x = \text{undefined or } \cos x = \frac{1}{2} \\
&\Rightarrow \underline{\underline{x = 60, 300}}.
\end{aligned}$$

(b) Solve

$$\cos^2 3y = 5 \sin^2 3y \text{ for } 0 \leq y \leq 2 \text{ radians.}$$

(4)

Solution

Well,

$$0 \leq y \leq 2 \Rightarrow 0 \leq 3y \leq 6$$

and

$$\begin{aligned}
\cos^2 3y = 5 \sin^2 3y &\Rightarrow \tan^2 3y = \frac{1}{5} \\
&\Rightarrow \tan 3y = \pm \frac{\sqrt{5}}{5}.
\end{aligned}$$

$$\underline{\tan x = \frac{\sqrt{5}}{5}} :$$

$$\begin{aligned}\tan 3y = \frac{\sqrt{5}}{5} &\Rightarrow 3y = 0.420\,534\,335\,5, 3.562\,126\,989 \text{ (FCD)} \\ &\Rightarrow y = 0.140\,178\,111\,8, 1.187\,375\,663 \text{ (FCD)} \\ &\Rightarrow \underline{y = 0.140, 1.19 \text{ (3 sf)}}.\end{aligned}$$

$$\underline{\tan x = -\frac{\sqrt{5}}{5}} :$$

$$\begin{aligned}\tan 3y = -\frac{\sqrt{5}}{5} &\Rightarrow 3y = 2.721\,058\,318, 5.862\,650\,972 \text{ (FCD)} \\ &\Rightarrow y = 0.907\,019\,439\,4, 1.954\,216\,911 \text{ (FCD)} \\ &\Rightarrow \underline{y = 0.907, 1.95 \text{ (3 sf)}}.\end{aligned}$$

(c) Solve

$$2 \operatorname{cosec}\left(z + \frac{1}{4}\pi\right) = 5 \text{ for } 0 \leq z \leq 6 \text{ radians.}$$

(4)

Solution

Well,

$$0 \leq z \leq 6 \Rightarrow 0.785 \dots \leq z + \frac{1}{4}\pi \leq 6.785 \dots$$

and

$$\begin{aligned}2 \operatorname{cosec}\left(z + \frac{1}{4}\pi\right) = 5 &\Rightarrow \operatorname{cosec}\left(z + \frac{1}{4}\pi\right) = \frac{5}{2} \\ &\Rightarrow \sin\left(z + \frac{1}{4}\pi\right) = \frac{2}{5} \\ &\Rightarrow z + \frac{1}{4}\pi = 2.730\,075\,808, 6.694\,702\,153 \text{ (FCD)} \\ &\Rightarrow z = 1.944\,677\,644, 5.909\,303\,99 \text{ (FCD)} \\ &\Rightarrow \underline{z = 1.94, 5.91 \text{ (3 sf)}}.\end{aligned}$$

EITHER

11. The tangent to the curve

$$y = 5e^x + 3e^{-x}$$

at the point where $x = \ln \frac{3}{5}$, meets the x -axis at the point P .

(a) Find the coordinates of P .

(5)

Solution

Well,

$$y = 5e^x + 3e^{-x} \Rightarrow \frac{dy}{dx} = 5e^x - 3e^{-x}$$

and

$$\begin{aligned} x = \ln \frac{3}{5} &\Rightarrow y = 5e^{\ln \frac{3}{5}} + 3e^{-\ln \frac{3}{5}} \\ &\Rightarrow y = 5e^{\ln \frac{3}{5}} + 3e^{\ln \frac{5}{3}} \\ &\Rightarrow y = 5\left(\frac{3}{5}\right) + 3\left(\frac{5}{3}\right) \\ &\Rightarrow y = 8. \end{aligned}$$

Now,

$$\begin{aligned} x = \ln \frac{3}{5} &\Rightarrow \frac{dy}{dx} = 5e^{\ln \frac{3}{5}} - 3e^{-\ln \frac{3}{5}} \\ &\Rightarrow \frac{dy}{dx} = 5e^{\ln \frac{3}{5}} - 3e^{\ln \frac{5}{3}} \\ &\Rightarrow \frac{dy}{dx} = 5\left(\frac{3}{5}\right) - 3\left(\frac{5}{3}\right) \\ &\Rightarrow \frac{dy}{dx} = -2. \end{aligned}$$

Next, the tangent is

$$y - 8 = -2\left(x - \ln \frac{3}{5}\right)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -8 = -2\left(x - \ln \frac{3}{5}\right) \\ &\Rightarrow 4 = x - \ln \frac{3}{5} \\ &\Rightarrow \underline{\underline{x = 4 + \ln \frac{3}{5}}}. \end{aligned}$$

The area of the region enclosed by the curve

$$y = 5e^x + 3e^{-x},$$

the y -axis, the positive x -axis, and the line $x = a$ is 12 square units.

(b) Show that

$$5e^{2a} - 14e^a - 3 = 0.$$

(3)

Solution

Well,

$$\begin{aligned}\int_0^a (5e^x + 3e^{-x}) dx &= 12 \Rightarrow [5e^x - 3e^{-x}]_{x=0}^a = 12 \\ &\Rightarrow (5e^a - 3e^{-a}) - (5 - 3) = 12 \\ &\Rightarrow 5e^a - 2 - 3e^{-a} = 12 \\ &\Rightarrow 5e^a - 14 - 3e^{-a} = 0\end{aligned}$$

multiply by e^a :

$$\Rightarrow \underline{\underline{5e^{2a} - 14e^a - 3 = 0}},$$

as required.

(c) Hence find the value of a .

(3)

Solution

$$\left. \begin{array}{l} \text{add to:} \quad \quad \quad -14 \\ \text{multiply to: } (+5) \times (-3) = -15 \end{array} \right\} -15, +1$$

Let $x = e^a$. E.g.,

$$\begin{aligned}5e^{2a} - 14e^a - 3 &= 0 \Rightarrow 5x^2 - 14x - 3 = 0 \\ &\Rightarrow 5x^2 - 15x + x - 3 = 0 \\ &\Rightarrow 5x(x - 3) + 1(x - 3) = 0 \\ &\Rightarrow (5x + 1)(x - 3) = 0 \\ &\Rightarrow x = -\frac{1}{5} \text{ or } x = 3 \\ &\Rightarrow e^a = -\frac{1}{5} \text{ (cannot happen) or } e^a = 3 \\ &\Rightarrow \underline{\underline{a = \ln 3}}.\end{aligned}$$

OR

12. (a) Given that

(4)

$$y = \frac{3e^{2x}}{1 + e^{2x}},$$

show that

$$\frac{dy}{dx} = \frac{Ae^{2x}}{(1 + e^{2x})^2},$$

where A is a constant to be found.

Solution

Quotient rule:

$$u = 3e^{2x} \Rightarrow \frac{du}{dx} = 6e^{2x}$$

$$v = 1 + e^{2x} \Rightarrow \frac{dv}{dx} = 2e^{2x}$$

and

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1 + e^{2x})(6e^{2x}) - (2e^{2x})(3e^{2x})}{(1 + e^{2x})^2} \\ &= \frac{6e^{2x} + 6e^{4x} - 6e^{4x}}{(1 + e^{2x})^2} \\ &= \frac{6e^{2x}}{(1 + e^{2x})^2}; \end{aligned}$$

hence, $A = 6$.

- (b) Find the equation of the tangent to the curve

(3)

$$y = \frac{3e^{2x}}{1 + e^{2x}}$$

at the point where the curve crosses the y -axis.

Solution

Well,

$$x = 0 \Rightarrow y = \frac{3}{2}$$

and

$$x = 0 \Rightarrow \frac{dy}{dx} = \frac{3}{2}.$$

Now, the equation of the tangent is

$$y - \frac{3}{2} = \frac{3}{2}(x - 0) \Rightarrow \underline{\underline{y = \frac{3}{2}x + \frac{3}{2}}}.$$

- (c) Using your result from part (a), find

(4)

$$\int \frac{e^{2x}}{(1 + e^{2x})^2} dx$$

and hence evaluate

$$\int_0^{\ln 3} \frac{e^{2x}}{(1 + e^{2x})^2} dx.$$

Solution

Well,

$$\begin{aligned} \int \frac{e^{2x}}{(1 + e^{2x})^2} dx &= \frac{1}{6} \int \frac{6e^{2x}}{(1 + e^{2x})^2} dx \\ &= \frac{1}{6} \cdot \frac{3e^{2x}}{1 + e^{2x}} + c \\ &= \frac{e^{2x}}{2(1 + e^{2x})} + c \end{aligned}$$

and

$$\begin{aligned} \int_0^{\ln 3} \frac{e^{2x}}{(1 + e^{2x})^2} dx &= \left[\frac{e^{2x}}{2(1 + e^{2x})} \right]_{x=0}^{\ln 3} \\ &= \frac{e^{2 \ln 3}}{2(1 + e^{2 \ln 3})} - \frac{1}{4} \\ &= \frac{e^{\ln 3^2}}{2(1 + e^{\ln 3^2})} - \frac{1}{4} \\ &= \frac{9}{2(1 + 9)} - \frac{1}{4} \\ &= \frac{9}{20} - \frac{1}{4} \\ &= \underline{\underline{\frac{1}{5}}}. \end{aligned}$$