Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 June Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Variables x and y are such that when $\sqrt{y}$ is plotted against x^2 a straight line graph passing through the points $(1, 3)$ and $(4, 18)$ is obtained. (4)

   Express y in terms of x.

Solution

Well,

$$\begin{aligned} m &= \frac{18-3}{4-1} \\ &= \frac{15}{3} \\ &= 5 \end{aligned}$$

and the equation is

$$\begin{aligned} \sqrt{y} - 3 = 5(x^2 - 1) &\Rightarrow \sqrt{y} - 3 = 5x^2 - 5 \\ &\Rightarrow \sqrt{y} = 5x^2 - 2 \\ &\Rightarrow \underline{\underline{y = (5x^2 - 2)^2}}. \end{aligned}$$

2. (a) Solve the equation (3)

$$3^{p+1} = 0.7,$$

   giving your answer to 2 decimal places.

Solution

Well,

$$\begin{aligned} 3^{p+1} = 0.7 &\Rightarrow p + 1 = \log_3 0.7 \\ &\Rightarrow p = \log_3 0.7 - 1 \\ &\Rightarrow p = -1.324\,659\,525 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{p = -1.32 \text{ (2 dp)}}}. \end{aligned}$$

(b) Express (3)

$$\frac{y \times (4x^3)^2}{\sqrt{8y^3}}$$

in the form

$$2^a \times x^b \times y^c,$$

where a, b, and c are constants.

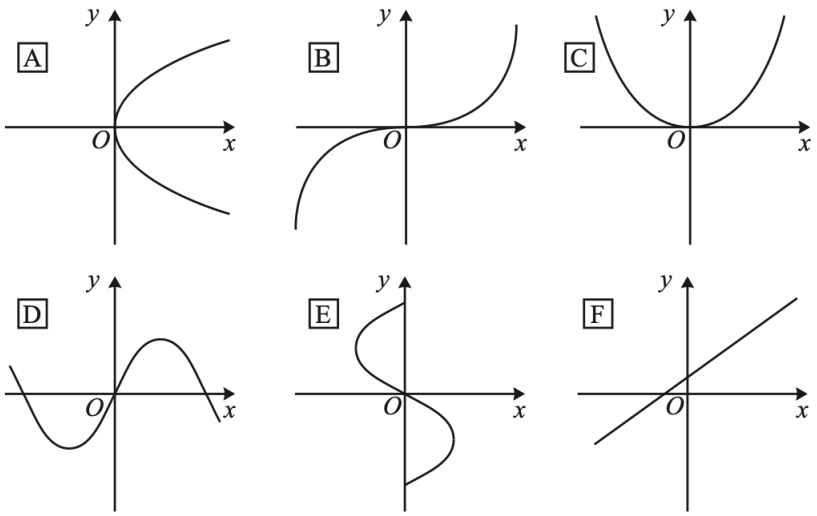
Solution

Now,

$$\begin{aligned} \frac{y \times (4x^3)^2}{\sqrt{8y^3}} &= \frac{y \times 16x^6}{(2^3)^{\frac{1}{2}} \times y^{\frac{3}{2}}} \\ &= \frac{y \times 2^4 \times x^6}{2^{\frac{3}{2}} \times y^{\frac{3}{2}}} \\ &= \underline{\underline{2^{\frac{5}{2}} \times x^6 \times y^{-\frac{1}{2}}}}; \end{aligned}$$

hence, $\underline{\underline{a = \frac{5}{2}}}$, $\underline{\underline{b = 6}}$, and $\underline{\underline{c = -\frac{1}{2}}}$.

3. (a) Here are 6 graphs.



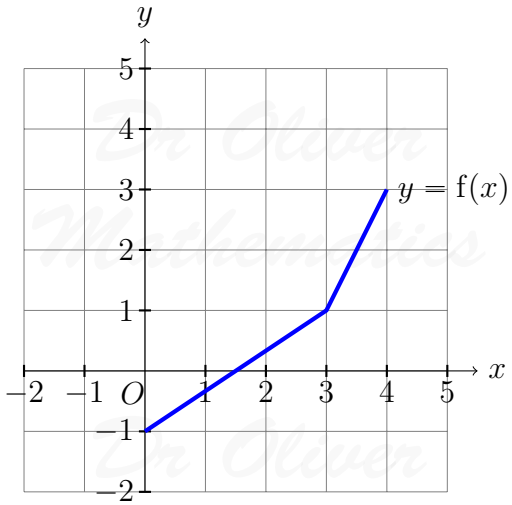
(i) Write down the letter of each graph which does **not** represent a function. (2)

Solution
$\underline{\underline{A}}$ and $\underline{\underline{E}}$.

(ii) Write down the letter of each graph which represents a function that does **not** have an inverse. (2)

Solution
$\underline{\underline{C}}$ and $\underline{\underline{E}}$.

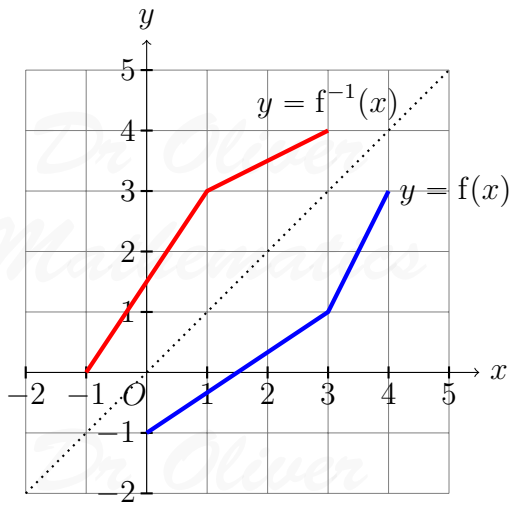
(b) The diagram shows the graph of a function $y = \mathrm{f}(x)$. (2)



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On the same axes sketch the graph on $y = \mathrm{f}^{-1}(x)$.

Solution



4. The position vectors of the points A and B, relative to an origin O, are $(4\mathbf{i} - 21\mathbf{j})$ and $(22\mathbf{i} - 30\mathbf{j})$ respectively.

   The point C lies on AB such that

$$\overrightarrow{AB} = 3\overrightarrow{AC}.$$

   (a) Find the position vector of C relative to O. (4)

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -(4\mathbf{i} - 21\mathbf{j}) + (22\mathbf{i} - 30\mathbf{j})\\ &= 18\mathbf{i} - 9\mathbf{j}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \overrightarrow{OA} + \tfrac{1}{3}\overrightarrow{AB}\\ &= (4\mathbf{i} - 21\mathbf{j}) + \tfrac{1}{3}(18\mathbf{i} - 9\mathbf{j})\\ &= \underline{\underline{10\mathbf{i} - 24\mathbf{j}}}.\end{aligned}$$

(b) Find the unit vector in the direction $\overrightarrow{OC}$. (2)

Solution

Well,

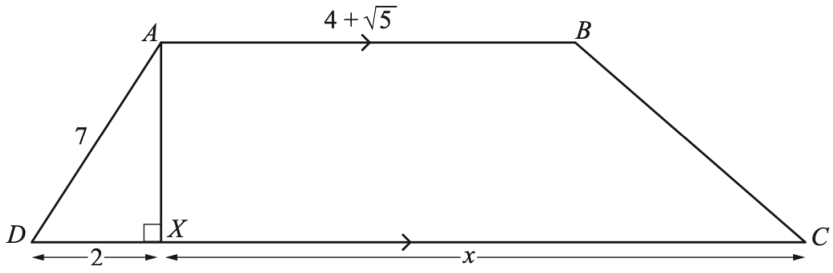
$$\begin{aligned} OC &= \sqrt{10^2 + (-24)^2} \\ &= 26 \end{aligned}$$

and the unit vector is

$$\tfrac{1}{26}(10\mathbf{i} - 24\mathbf{j}) = \underline{\underline{\tfrac{1}{13}(5\mathbf{i} - 12\mathbf{j})}}.$$

5. **Calculators must not be used in this question.** (6)

The diagram shows a trapezium $ABCD$ in which $AD = 7$ cm and $AB = (4 + \sqrt{5})$ cm.



AX is perpendicular to DC with $DX = 2$ cm and $XC = x$ cm.

Given that the area of trapezium $ABCD$ is

$$15(\sqrt{5} + 2) \text{ cm}^2,$$

obtain an expression for x in the form $a + b\sqrt{5}$, where a and b are integers.

Solution

Pythagoras' theorem:

$$\begin{aligned} AD^2 = AX^2 + XD^2 &\Rightarrow 7^2 = AX^2 + 2^2 \\ &\Rightarrow AX^2 = 45 \\ &\Rightarrow AX = 3\sqrt{5}. \end{aligned}$$

Now,

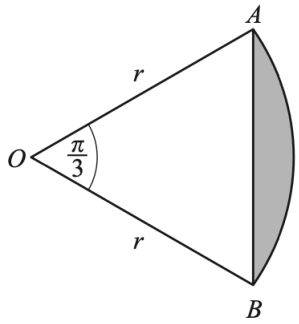
$$\begin{aligned}\text{area} &= \tfrac{1}{2} \times 3\sqrt{5} \times [(x+2) + (4+\sqrt{5})] \\ &= \tfrac{3}{2}\sqrt{5}(x+6+\sqrt{5}).\end{aligned}$$

The two expressions are equal:

$$\begin{aligned}\tfrac{3}{2}\sqrt{5}(x+6+\sqrt{5}) = 15(\sqrt{5}+2) &\Rightarrow \tfrac{3}{2}\sqrt{5}(x+6+\sqrt{5}) = 15\sqrt{5}+30 \\ &\Rightarrow \tfrac{3}{2}\sqrt{5}(x+6+\sqrt{5}) = 15\sqrt{5}+30 \\ &\Rightarrow \sqrt{5}(x+6+\sqrt{5}) = 10\sqrt{5}+20 \\ &\Rightarrow \sqrt{5}(x+6)+5 = 10\sqrt{5}+20 \\ &\Rightarrow \sqrt{5}(x+6)+5 = 10\sqrt{5}+15 \\ &\Rightarrow x+6 = \frac{10\sqrt{5}+15}{\sqrt{5}} \\ &\Rightarrow x+6 = \frac{10\sqrt{5}+15}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &\Rightarrow x+6 = \frac{50+15\sqrt{5}}{5} \\ &\Rightarrow x = \frac{50+15\sqrt{5}}{5} - 6 \\ &\Rightarrow x = \frac{50+15\sqrt{5}-30}{5} \\ &\Rightarrow x = \frac{20+15\sqrt{5}}{5} \\ &\Rightarrow \underline{\underline{x = 4+3\sqrt{5}}};\end{aligned}$$

hence, $\underline{\underline{a = 4}}$ and $\underline{\underline{b = 3}}$.

6. The shaded region in the diagram is a segment of a circle with centre O and radius r cm.



Angle $AOB = \frac{1}{3}\pi$ radians.

(a) Show that the perimeter of the segment is (2)

$$r\left(\frac{3+\pi}{3}\right) \text{ cm.}$$

Solution

Well, $\triangle OAB$ is a equilateral triangle (why?) so

$$\begin{aligned} \text{perimeter of the segment} &= AB + \text{curve } AB \\ &= r + \tfrac{1}{3}r\pi \\ &= r(1 + \tfrac{1}{3}\pi) \\ &= \underline{\underline{r\left(\frac{3+\pi}{3}\right)}}, \end{aligned}$$

as required.

(b) Given that the perimeter of the segment is 26 cm, find the value of r and the area of the segment. (5)

Solution

Now,

$$r\left(\frac{3+\pi}{3}\right) = 26 \Rightarrow \underline{\underline{r = \left(\frac{78}{3+\pi}\right) \text{ cm}}}$$

$$\Rightarrow \underline{\underline{r = 12.7 \text{ cm (3 sf)}}}.$$

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Next,

$$\begin{aligned}\text{segment} &= \text{sector} - \text{triangle}\\ &= \left[\tfrac{1}{2} \times \left(\frac{78}{3+\pi}\right)^2 \times \tfrac{1}{3}\pi\right] - \left[\tfrac{1}{2} \times \left(\frac{78}{3+\pi}\right)^2 \times \sin(\tfrac{1}{3}\pi)\right]\\ &= 14.611\,292\,18 \text{ (FCD)}\\ &= \underline{\underline{14.6 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

7. Differentiate, with respect to x,

(a) $(3-5x)^{12}$, (2)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12(3-5x)^{11} \times (-5)\\ &= \underline{\underline{-60(3-5x)^{11}}}.\end{aligned}$$

(b) $x^2 \sin x$, (2)

Solution

Product rule:

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$v = \sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2)(\cos x) + (2x)(\sin x)\\ &= \underline{\underline{x^2 \cos x + 2x \sin x}}.\end{aligned}$$

(c) $\dfrac{\tan x}{1+\mathrm{e}^{2x}}$. (4)

Solution

Quotient rule:

$$u = \tan x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$$
$$v = 1 + \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(1+\mathrm{e}^{2x})(\sec^2 x) - (2\mathrm{e}^{2x})(\tan x)}{(1+\mathrm{e}^{2x})^2} \\ &= \underline{\underline{\frac{\sec^2 x(1+\mathrm{e}^{2x}) - 2\mathrm{e}^{2x}\tan x}{(1+\mathrm{e}^{2x})^2}}}.\end{aligned}$$

8. **Solutions to this question by accurate drawing will not be accepted.**
The points $A(-6, 2)$, $B(2, 6)$, and C are the vertices of a triangle.

(a)) Find the equation of the line AB in the form $y = mx + c$. (2)

Solution

Well,

$$\begin{aligned}m_{AB} &= \frac{6-2}{2-(-6)} \\ &= \frac{4}{8} \\ &= \tfrac{1}{2}\end{aligned}$$

and the equation is

$$\begin{aligned}y - 6 = \tfrac{1}{2}(x-2) &\Rightarrow y - 6 = \tfrac{1}{2}x - 1 \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{2}x + 5}}.\end{aligned}$$

(b) Given that angle $ABC = 90°$, find the equation of BC. (2)

Solution

Now,

$$m_{BC} = -2$$

and the equation of BC is

$$y - 6 = -2(x - 2) \Rightarrow y - 6 = -2x + 4$$
$$\Rightarrow \underline{\underline{y = -2x + 10}}.$$

(c) Given that the length of AC is 10 units, find the coordinates of each of the two possible positions of point C. (4)

Solution

Now,

$$(x + 6)^2 + (y - 2)^2 = 10^2 \Rightarrow (x + 6)^2 + (-2x + 10 - 2)^2 = 100$$
$$\Rightarrow (x + 6)^2 + (-2x + 8)^2 = 100$$

$\times$	x	$+6$
x	x^2	$+6x$
$+6$	$+6x$	$+36$

$\times$	$-2x$	$+8$
$-2x$	$4x^2$	$-16x$
$+8$	$-16x$	$+64$

$$\Rightarrow (x^2 + 12x + 36) + (4x^2 - 32x + 64) = 100$$
$$\Rightarrow 5x^2 - 20x = 0$$
$$\Rightarrow 5x(x - 4) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 4$$
$$\Rightarrow y = 10 \text{ or } y = 2;$$

hence, the coordinates are $\underline{\underline{(0, 10)}}$ and $\underline{\underline{(4, 2)}}$.

9. (a) The graph of (3)

$$y = k(3^x) + c$$

passes through the points $(0, 14)$ and $(-2, 6)$.

Find the value of k and of c.

Solution
Well,

$$x = 0, y = 14 \Rightarrow 14 = k + c \quad (1)$$
$$x = -2, y = 6 \Rightarrow 6 = \tfrac{1}{9}k + c \quad (2).$$

Do (1) − (2):

$$\tfrac{8}{9}k = 8 \Rightarrow \underline{\underline{k = 9}}$$
$$\Rightarrow \underline{\underline{c = 5}}.$$

(b) The variables x and y are connected by the equation

$$y = \mathrm{e}^x + 25 - 24\mathrm{e}^{-x}.$$

(i) Find the value of y when $x = 4$. (1)

Solution
Well,

$$x = 4 \Rightarrow y = \mathrm{e}^4 + 25 - 24\mathrm{e}^{-4}$$
$$\Rightarrow y = 79.158\,574\,7 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{y = 79.2 \text{ (3 sf)}}}.$$

(ii) Find the value of e^x when $y = 20$ and hence find the corresponding value of x. (4)

Solution
Now,

$$y = 20 \Rightarrow \mathrm{e}^x + 25 - 24\mathrm{e}^{-x} = 20$$
$$\Rightarrow \mathrm{e}^x + 5 - 24\mathrm{e}^{-x} = 20$$

multiply by e^x:

$$\Rightarrow \mathrm{e}^{2x} + 5\mathrm{e}^x - 24 = 0$$

add to: $+5$
multiply to: -24
$\Big\}$ $-3,\ +8$

$\Rightarrow (\mathrm{e}^x - 3)(\mathrm{e}^x + 8) = 0$

$\Rightarrow \mathrm{e}^x - 3 = 0$ (as $\mathrm{e}^x + 8 = 0$ has no solutions)

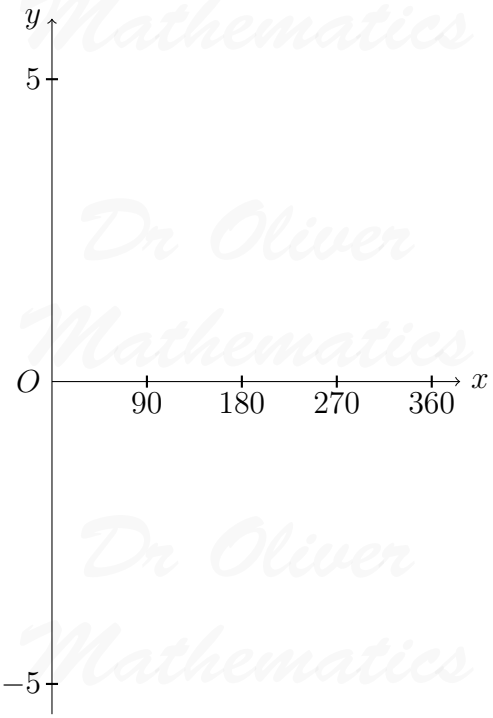
$\Rightarrow \underline{\underline{\mathrm{e}^x = 3}}$

$\Rightarrow \underline{\underline{x = \ln 3}}$.

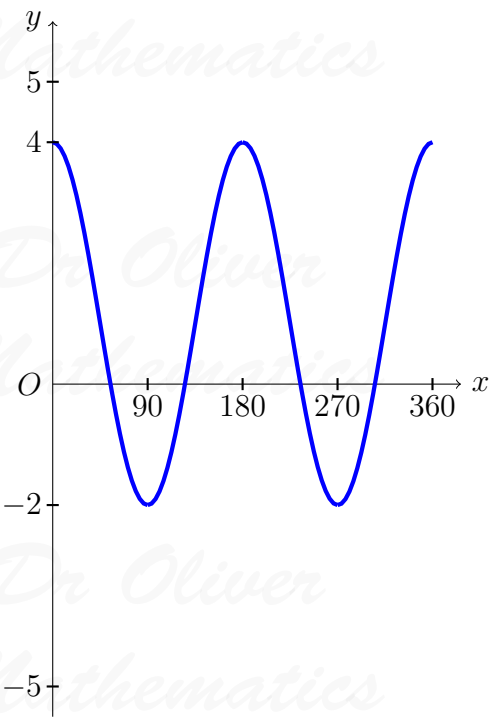
10. (a) The function f is defined, for $0° \leqslant x \leqslant 360°$, by

$$\mathrm{f}(x) = 1 + 3\cos 2x.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$ on the axes below. (4)



Solution



(ii) State the amplitude of f. (1)

Solution

$\underline{\underline{3}}$.

(iii) State the period of f. (1)

Solution

$$\frac{360}{2} = \underline{\underline{\underline{180}}}.$$

(b) Given that (3)

$$\cos x = p,$$

where $270° < x < 360°$, find $\operatorname{cosec} x$ in terms of p.

Solution

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$$\begin{aligned}\cos x = p &\Rightarrow \cos^2 x = p \\ &\Rightarrow 1 - \cos^2 x = 1 - p^2 \\ &\Rightarrow \sin^2 x = 1 - p^2 \\ &\Rightarrow \sin x = -\sqrt{1-p^2} \\ &\Rightarrow \operatorname{cosec} x = -\frac{1}{\sqrt{1-p^2}},\end{aligned}$$

because $\sin x$ is negative for $270^\circ < x < 360^\circ$.

11. A curve has equation

$$y = 3x + \frac{1}{(x-4)^3}.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (4)

Solution

Well,

$$\begin{aligned}y = 3x + \frac{1}{(x-4)^3} &\Rightarrow y = 3x + (x-4)^{-3} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 3(x-4)^{-4}}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12(x-4)^{-5}}}.\end{aligned}$$

(b) Show that the coordinates of the stationary points of the curve are $(5, 16)$ and $(3, 8)$. (2)

Solution

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Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3 - 3(x-4)^{-4} = 0\\ &\Rightarrow 3 = 3(x-4)^{-4}\\ &\Rightarrow 1 = (x-4)^{-4}\\ &\Rightarrow (x-4)^4 = 1\\ &\Rightarrow x - 4 = 1 \text{ or } x - 4 = -1\\ &\Rightarrow x = 5 \text{ or } x = 3\\ &\Rightarrow y = 16 \text{ or } y = 8;\end{aligned}$$

hence, the coordinates of the stationary points of the curve are $\underline{\underline{(5, 16)}}$ and $\underline{\underline{(3, 8)}}$.

(c) Determine the nature of each of these stationary points. (2)

Solution

Next,

$$\begin{aligned}x = 3 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -12 < 0\\ x = 5 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12 > 0;\end{aligned}$$

hence, $(3, 8)$ is $\underline{\underline{\text{maximum}}}$ but $(5, 16)$ is $\underline{\underline{\text{minimum}}}$.

(d) Find (2)

$$\int \left(3x + \frac{1}{(x-4)^3}\right) \mathrm{d}x.$$

Solution

Well,

$$\begin{aligned}\int \left(3x + \frac{1}{(x-4)^3}\right) \mathrm{d}x &= \int \left[3x + (x-4)^{-3}\right] \mathrm{d}x\\ &= \underline{\underline{\tfrac{3}{2}x^2 - \tfrac{1}{2}(x-4)^{-2} + c}}.\end{aligned}$$

(e) Hence find the area of the region enclosed by the curve, the line $x = 5$, the x-axis, and the line $x = 6$. (2)

Solution

Now,

$$\begin{aligned}\int_5^6 \left(3x + \frac{1}{(x-4)^3}\right) \mathrm{d}x &= \left[\tfrac{3}{2}x^2 - \tfrac{1}{2}(x-4)^{-2}\right]_{x=5}^{6} \\ &= \left(54 - \tfrac{1}{8}\right) - \left(37\tfrac{1}{2} - \tfrac{1}{2}\right) \\ &= \underline{\underline{16\tfrac{7}{8}}}.\end{aligned}$$

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