

Dr Oliver Mathematics
Further Mathematics: Core Pure Mathematics 2
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.

You must write down all the stages in your working.

Inexact answers should be given to three significant figures unless otherwise stated.

1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3i$ and $z_2 = \sqrt{3} + i$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$.

The student's attempt is shown below.

Line 1 \rightarrow $\arg(z_1) = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

Line 2 \rightarrow $\arg(z_2) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Line 3 \rightarrow $\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$

Line 4 \rightarrow $= \frac{\left(\frac{\pi}{4}\right)}{\left(\frac{\pi}{6}\right)} = \frac{3}{2}$

The student made errors in line 1 and line 3.

Correct the error that the student made in

- (a) (i) line 1, (2)
(ii) line 3.

- (b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$. (1)

2. In this question you must show all stages of your working.

A college offers only three courses: Construction, Design, and Hospitality.

Each student enrolls on just one of these courses.

In 2019, there was a total of 1 110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction increased by 1.25%,
- Design increased by 2.5%, and
- Hospitality decreased by 2%.

In 2020, the total number of students at the college increased by 0.27%, to 2 significant figures.

- (a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019. (4)
- (ii) Using your variables from part (a)(i), write down **three** equations that model this situation.
- (b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019. (4)

3.

$$\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}, \text{ where } a \text{ is a constant.}$$

- (a) Prove by mathematical induction that, for $n \in \mathbb{N}$, (6)

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{1}{2}a(3^n - 1) \\ 0 & 1 \end{pmatrix}.$$

Triangle T has vertices A , B , and C .

Triangle T is transformed to triangle T' by the transformation represented by \mathbf{M}^n , where $n \in \mathbb{N}$.

Given that

- triangle T has an area of 5 cm^2 ,
- triangle T' has an area of $1\,215 \text{ cm}^2$, and
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$,

- (b) determine (5)

- (i) the value of n ,
(ii) the value of a .

4. (a) Given that

$$z_1 = 6e^{\frac{1}{3}\pi i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5}{6}\pi i},$$

show that

$$z_1 + z_2 = 12e^{\frac{2}{3}\pi i}.$$

(b) Given that

$$\arg(z - 5) = \frac{2}{3}\pi,$$

determine the least value of $|z|$ as z varies.

5. (a) Given that show that

$$y = \arcsin x, \quad -1 \leq x \leq 1,$$

show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$f(x) = \arcsin(e^x), \quad x \leq 0.$$

(b) Prove that $f(x)$ has no stationary points.

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0,$$

where p and q are real positive constants, has roots α , β , and γ .

Given that

$$\alpha^2 + \beta^2 + \gamma^2 = 16,$$

(a) show that $p = 12$.

Given that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3},$$

(b) determine the value of q

Without solving the cubic equation,

(c) determine the value of

$$(\alpha - 1)(\beta - 1)(\gamma - 1).$$

7. Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan \theta, \quad 0 \leq \theta < \frac{1}{3}\pi.$$

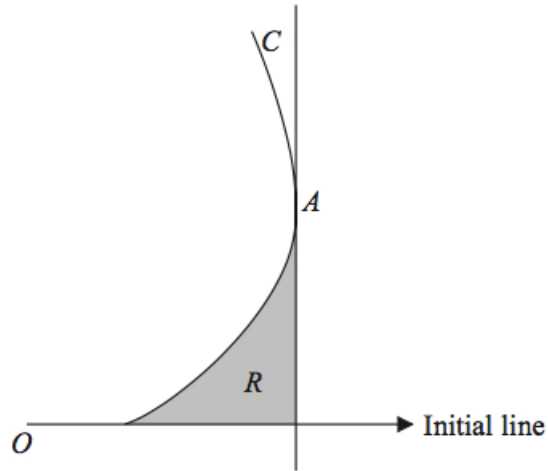


Figure 1: $r = 1 + \tan \theta$

Figure 1 also shows the tangent to C at the point A . This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $(2, \frac{1}{4}\pi)$. (4)

The finite region R , shown shaded in Figure 1, is bounded by C , the tangent at A , and the initial line.

(b) Use calculus to show that the exact area of R is (6)

$$\frac{1}{2}(1 - \ln 2).$$

8. Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120° .

(a) Determine the value of a . (4)

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point. (5)

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2.$$

(c) Hence determine the shortest distance from the nest to the ground level of the park. (3)

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer. (1)

9.

$$y = \cosh^n x, \quad n \geq 5.$$

(a) (i) Show that (4)

$$\frac{d^2 y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x.$$

(ii) Determine an expression for (2)

$$\frac{d^4 y}{dx^4}.$$

(b) Hence determine the first three non-zero terms of the Maclaurin series for y , giving each coefficient in simplest form. (2)