Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2015 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Given that (3)

$$y = \mathrm{e}^{5x} \tan 2x,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$u = \mathrm{e}^{5x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 5\mathrm{e}^{5x}$$
$$v = \tan 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\sec^2 2x$$

$$y = \mathrm{e}^{5x} \tan 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (\mathrm{e}^{5x})(2\sec^2 2x) + (5\mathrm{e}^{5x})(\tan 2x)$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{5x}(2\sec^2 2x + 5\tan 2x)}}.$$

2. (a) Given matrix (2)

$$\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix},$$

find $\mathbf{A}^2$ and show that the inverse of $\mathbf{A}^2$ exists.

Solution

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}}}$$

and

$$\begin{aligned}\det \mathbf{A}^2 &= -16 + 20 \\ &= 4\end{aligned}$$

so $\mathbf{A}^2$ does exist.

(b) Hence, or otherwise, find matrix $\mathbf{B}$ such that (3)

$$\mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}.$$

Solution

$$\begin{aligned}\mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} &\Rightarrow \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \tfrac{1}{4}\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \tfrac{1}{4}\begin{pmatrix} 4 & -44 \\ 0 & -20 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & -11 \\ 0 & -5 \end{pmatrix}.\end{aligned}$$

3. A curve is defined by (5)

$$y = \frac{\sin x}{2 - \cos x} \text{ for } 0 \leqslant x \leqslant \pi.$$

Find the exact values of the coordinates of the stationary point of this curve.

Solution

$$\begin{aligned}u = \sin x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \\ v = 2 - \cos x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x\end{aligned}$$

$$\begin{aligned} y = \frac{\sin x}{2 - \cos x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2 - \cos x)(\cos x) - (\sin x)(\sin x)}{(2 - \cos x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2 - \cos x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos x - 1}{(2 - \cos x)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{2\cos x - 1}{(2 - \cos x)^2} = 0 \\ &\Rightarrow 2\cos x - 1 = 0 \\ &\Rightarrow \cos x = \tfrac{1}{2} \\ &\Rightarrow x = \tfrac{1}{3}\pi \text{ (only)} \\ &\Rightarrow y = \tfrac{\sqrt{3}}{3}; \end{aligned}$$

hence, the exact values of the coordinate are $\underline{\underline{(\tfrac{1}{3}\pi, \tfrac{\sqrt{3}}{3})}}$.

4. (a) Express (1)

$$\log_a 2 + \log_a 4 + \log_a 8$$

in the form

$$p \log_a 2,$$

where p is a constant.

Solution

$$\begin{aligned} \log_a 2 + \log_a 4 + \log_a 8 &= \log_a 2 + \log_a 2^2 + \log_a 2^3 \\ &= \log_a 2 + 2\log_a 2 + 3\log_a 2 \\ &= \underline{\underline{6\log_a 2}}; \end{aligned}$$

hence, $\underline{\underline{p = 6}}$.

(b) Hence evaluate (3)

$$\sum_{r=1}^{100} \log_a 2^r,$$

giving your answer in the form

$$q \log_a 2,$$

where q is a constant.

Solution

$$\begin{aligned}\sum_{r=1}^{100} \log_a 2^r &= \sum_{r=1}^{100} r \log_a 2 \\ &= \log_a 2 \sum_{r=1}^{100} r \\ &= \log_a 2 \cdot \tfrac{1}{2}(100)(101) \\ &= \underline{\underline{5\,050 \log_a 2}};\end{aligned}$$

hence, $\underline{\underline{q = 5\,050}}$.

5. Find the general solution, in the form $y = \mathrm{f}(x)$, of the differential equation (6)

$$\frac{1}{\cos x} \frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \tan x,\ 0 < x < \pi.$$

Solution

$$\begin{aligned}\frac{1}{\cos x} \frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \tan x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y \cos x \tan x = \cos x \tan x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \sin x\end{aligned}$$

$$\begin{aligned}\text{IF} &= e^{\int \sin x \,dx} \\ &= e^{-\cos x}\end{aligned}$$

$$\begin{aligned}&\Rightarrow e^{-\cos x}\frac{dy}{dx} + ye^{-\cos x}\sin x = e^{-\cos x}\sin x \\ &\Rightarrow \frac{d}{dx}(e^{-\cos x}y) = \sin x\, e^{-\cos x} \\ &\Rightarrow e^{-\cos x}y = \int \sin x\, e^{-\cos x}\, dx \\ &\Rightarrow e^{-\cos x}y = e^{-\cos x} + c \\ &\Rightarrow \underline{\underline{y = 1 + c\, e^{\cos x}}}.\end{aligned}$$

6. (a) Express (3)

$$\frac{1}{1-y^2}$$

in partial fractions.

Solution

$$\left.\begin{array}{ll}\text{add to:} & 0 \\ \text{multiply to:} & -1\end{array}\right\} -1,\ +1$$

$$\begin{aligned}\frac{1}{1-y^2} &\equiv \frac{1}{(1+y)(1-y)} \\ &\equiv \frac{A}{1+y} + \frac{B}{1-y} \\ &\equiv \frac{A(1-y)+B(1+y)}{(1+y)(1-y)}\end{aligned}$$

which means

$$1 \equiv A(1-y) + B(1+y).$$

$\underline{y=1}$: $1 = 2B \Rightarrow B = \frac{1}{2}$.
$\underline{y=-1}$: $1 = 2A \Rightarrow A = \frac{1}{2}$.
Hence,

$$\frac{1}{1-y^2} \equiv \underline{\underline{\frac{\frac{1}{2}}{1+y} + \frac{\frac{1}{2}}{1-y}}}.$$

(b) Use the substitution $u = \sqrt{1-x}$ to obtain (6)

$$\int \frac{1}{x\sqrt{1-x}}\,\mathrm{d}x,\ 0 < x < 1.$$

Solution

$$\begin{aligned} u = \sqrt{1-x} &\Rightarrow u = (1-x)^{\frac{1}{2}} \\ &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\tfrac{1}{2}(1-x)^{-\frac{1}{2}} \\ &\Rightarrow \mathrm{d}u = -\tfrac{1}{2}(1-x)^{-\frac{1}{2}}\,\mathrm{d}x \\ &\Rightarrow \mathrm{d}u = -\frac{1}{2\sqrt{1-x}}\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} u = \sqrt{1-x} &\Rightarrow u^2 = 1-x \\ &\Rightarrow x = 1-u^2. \end{aligned}$$

Finally,

$$\begin{aligned} \int \frac{1}{x\sqrt{1-x}}\,\mathrm{d}x &= \int \frac{2}{x(2\sqrt{1-x})}\,\mathrm{d}x \\ &= \int \frac{-2}{1-u^2}\,\mathrm{d}u \\ &= \int -2\left(\frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\right)\mathrm{d}u \\ &= \int \left(-\frac{1}{1+u} - \frac{1}{1-u}\right)\mathrm{d}u \\ &= -\ln(1+u) + \ln(1-u) + c \\ &= \ln\left(\frac{1-u}{1+u}\right) + c \\ &= \underline{\underline{\ln\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right) + c.}} \end{aligned}$$