Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2011 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Differentiate the following, simplifying where possible.

   (a) $\text{f}(x) = \dfrac{1 + \sin x}{1 + 2\sin x}$, $0 \leqslant x \leqslant \pi$, (3)

   Solution

   $$u = 1 + \sin x \Rightarrow \frac{\text{d}u}{\text{d}x} = \cos x$$
   $$v = 1 + 2\sin x \Rightarrow \frac{\text{d}v}{\text{d}x} = 2\cos x$$

   $$\begin{aligned} \text{f}(x) = \frac{1 + \sin x}{1 + 2\sin x} &\Rightarrow \text{f}'(x) = \frac{(1 + 2\sin x)(\cos x) - (1 + \sin x)(2\cos x)}{(1 + 2\sin x)^2} \\ &\Rightarrow \text{f}'(x) = \frac{\cos x[(1 + 2\sin x) - (2 + 2\sin x)]}{(1 + 2\sin x)^2} \\ &\Rightarrow \text{f}'(x) = \frac{-\cos x}{(1 + 2\sin x)^2}. \end{aligned}$$

   (b) $\text{g}(x) = \ln(1 + \text{e}^{2x})$. (2)

   Solution

   $$\begin{aligned} \text{g}(x) = \ln(1 + \text{e}^{2x}) &\Rightarrow \text{g}'(x) = \frac{1}{1 + \text{e}^{2x}} \cdot \text{e}^{2x} \cdot 2 \\ &\Rightarrow \text{g}'(x) = \frac{2\text{e}^{2x}}{1 + \text{e}^{2x}}. \end{aligned}$$

2. (a) Given (2)

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix},$$

obtain $\mathbf{A}^{-1}$.

Solution

$$\det \mathbf{A} = 0 - (-6) = 6$$

and so

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}}}.$$

(b) Given (3)

$$\mathbf{AB} = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix},$$

find the matrix $\mathbf{B}$.

Solution

$$\begin{aligned} & \mathbf{AB} = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\ \Rightarrow\quad & \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\ \Rightarrow\quad & \mathbf{B} = \tfrac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\ \Rightarrow\quad & \mathbf{B} = \tfrac{1}{6}\begin{pmatrix} 12 & -6 \\ 18 & 6 \end{pmatrix} \\ \Rightarrow\quad & \mathbf{B} = \underline{\underline{\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}}}. \end{aligned}$$

3. A curve is defined by the equations (4)

$$x = 5\cos t \text{ and } y = 3\sin t,\ 0 \leqslant t < 2\pi.$$

Find the gradient of the curve when $t = \frac{1}{6}\pi$.

Solution

$$x = 5\cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -5\sin t$$
$$y = 3\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 3\cos t.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{3\cos t}{-5\sin t}$$

and

$$t = \tfrac{1}{6}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{-\tfrac{3}{5}\sqrt{3}}}.$$

4. (a) Find the value of N for which (3)

$$\sum_{r=1}^{N} r = 210.$$

Solution

$$\begin{aligned}\sum_{r=1}^{N} r = 210 &\Rightarrow \tfrac{1}{2}N(N+1) = 210\\ &\Rightarrow N(N+1) = 420\\ &\Rightarrow N^2 + N - 420 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +1\\ \text{multiply to:} & -420\end{array}\right\} -20,\ +21$$

$$\begin{aligned}&\Rightarrow (N-20)(N+21) = 0\\ &\Rightarrow N-20 = 0 \text{ or } N+21 = 0\\ &\Rightarrow N = 20 \text{ or } N = -21;\end{aligned}$$

as $N > 0$, $\underline{\underline{N = 20}}$.

(b) Evaluate (2)

$$\sum_{r=1}^{N} r^2$$

for this value of N.

Solution

$$\begin{aligned}\sum_{r=1}^{20} r^2 &= \tfrac{1}{6}(20)(20+1)(2\cdot 20+1)\\ &= \tfrac{1}{6}(20)(21)(41)\\ &= \underline{\underline{2\,870}}.\end{aligned}$$

5. Use the substitution $u = \ln x$ to obtain (4)

$$\int \frac{2}{x \ln x}\,\mathrm{d}x,$$

where $x > 1$.

Solution

$$\begin{aligned}u = \ln x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}\\ &\Rightarrow \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int \frac{2}{x \ln x}\,\mathrm{d}x &= \int \frac{2}{u}\,\mathrm{d}u\\ &= 2\ln u + c\\ &= \underline{\underline{2\ln(\ln x) + c}}.\end{aligned}$$

6. At any point (x, y) on a curve C, where $x \neq 0$, the gradient of the tangent is (9)

$$4 - \frac{3y}{x}.$$

Given that the point $(1, 3)$ lies on C, obtain an equation for C in the form $y = \mathrm{f}(x)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - \frac{3y}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3y}{x} = 4$$

$$\begin{aligned}\text{IF} &= \mathrm{e}^{\int \frac{3}{x}\,\mathrm{d}x} \\ &= \mathrm{e}^{3\ln x} \\ &= \mathrm{e}^{\ln x^3} \\ &= x^3\end{aligned}$$

$$\begin{aligned}&\Rightarrow x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2y = 4x^3 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^3y) = 4x^3 \\ &\Rightarrow x^3y = \int 4x^3\,\mathrm{d}x \\ &\Rightarrow x^3y = x^4 + c.\end{aligned}$$

Now,

$$x = 1, y = 3 \Rightarrow 3 = 1 + c \Rightarrow c = 2$$

and, finally,

$$x^3y = x^4 + 2 \Rightarrow \underline{\underline{y = x + \frac{2}{x^3}}}.$$