Dr Oliver Mathematics
Mathematics
Differentiation Part 3
Past Examination Questions

This booklet consists of 53 questions across a variety of examination topics.
The total number of marks available is 439.

1. (a) Differentiate with respect to x:
   (i) $3\sin^2 x + \sec 2x$, (3)
   (ii) $[x + \ln(2x)]^3$. (3)

   Given that $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$, $x \neq 1$,

   (b) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{8}{(x-1)^3}$. (6)

2. (3)
$$\mathrm{f}(x) = 3\mathrm{e}^x - \tfrac{1}{2}\ln x - 2,\ x > 0.$$
   Differentiate to find $\mathrm{f}'(x)$.

3. (a) Differentiate with respect to x:
   (i) $x^2\mathrm{e}^{3x+2}$, (4)
   (ii) $\dfrac{\cos(2x^3)}{3x}$. (4)

   (b) Given that $x = 4\sin(2y + 6)$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (5)

4. Differentiate with respect to x:
   (a) $\mathrm{e}^{3x} + \ln 2x$, (3)
   (b) $(5 + x^2)^{\frac{3}{2}}$. (3)

5. The curve
$$y = (2x - 1)\tan 2x,\ 0 \leqslant x < \tfrac{\pi}{4},$$
   has a minimum at the point P. The x-coordinate of P is k.

   Show that k satisfies the equation
$$4k + \sin 4k - 2 = 0.$$

6. The curve C has equation $x = 2\sin y$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ at $P\left(\sqrt{2}, \dfrac{\pi}{4}\right)$. (4)

(b) Find an equation of the normal to C at P. Give your answer in the form $y = mx + c$, where m and c are exact constants. (4)

7. (a) The curve C has equation (6)

$$y = \frac{x}{9 + x^2}.$$

Use calculus to find the coordinates of the turning points of C.

(b) Given that (5)

$$y = (1 + \mathrm{e}^{2x})^{\frac{3}{2}},$$

find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{2}\ln 3$.

8.

$$\mathrm{f}(x) = \frac{2x+3}{x+2} - \frac{9+2x}{2x^2+3x-2}, \; x > \tfrac{1}{2}.$$

(a) Show that $\mathrm{f}(x) = \dfrac{4x-6}{2x-1}$. (7)

(b) Hence, or otherwise, find $\mathrm{f}'(x)$ in its simplest form. (3)

9. A curve C has equation

$$y = x^2\mathrm{e}^x.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, using the product rule for differenitation. (3)

(b) Hence find the coordinates of the turning points of C. (3)

(c) Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (2)

(d) Determine the nature of each turning point of the curve C. (2)

10. (a) A curve C has equation (6)

$$y = \mathrm{e}^{2x}\tan x, \; x \neq (2n+1)\frac{\pi}{2}.$$

Show that the turning points on C where $\tan x = -1$.

(b) Find an equation of the tangent to C at the point where $x = 0$. (2)

11. The point P lies on the curve with equation

$$y = 4\mathrm{e}^{2x+1}.$$

The y-coordinate of P is 8.

(a) Find, in terms of $\ln 2$, the x-coordinate of P. (2)

(b) Find an equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found. (4)

12. (a) Differentiate with respect to x:

(i) $e^{3x}(\sin x + 2\cos x)$, (3)

(ii) $x^3 \ln(5x + 2)$. (3)

Given that $y = \dfrac{3x^2 + 6x - 7}{(x+1)^2}$, $x \neq -1$,,

(b) show that $\dfrac{dy}{dx} = \dfrac{20}{(x+1)^3}$. (5)

(c) Hence find $\dfrac{d^2y}{dx^2}$ and the real values of x for which $\dfrac{d^2y}{dx^2} = -\dfrac{15}{4}$. (3)

13. Find the equation of the tangent to the curve $x = \cos(2y + \pi)$ at $\left(0, \frac{\pi}{4}\right)$. (6)

Give your answer in the form $y = ax + b$, where a and b are constant to be found.

14. (a) Find the value of $\dfrac{dy}{dx}$ at the point where $x = 2$ on the curve with equation (6)

$$y = x^2\sqrt{5x - 1}.$$

(b) Differentiate $\dfrac{\sin 2x}{x^2}$ with respect to x. (4)

15. (5)

$$f(x) = 3xe^x - 1.$$

The curve with equation $y = f(x)$ has a turning point P.

Find the exact coordinates of P.

16.

$$f(x) = \frac{2x + 2}{x^2 - 2x - 3} - \frac{x + 1}{x - 3}.$$

(a) Express $f(x)$ as a single fraction in its simplest form. (4)

(b) Hence show that $f'(x) = \dfrac{2}{(x-3)^2}$. (3)

17. (a) Differentiate with respect to x:

(i) $x^2 \cos 3x$, (3)

(ii) $\dfrac{\ln(x^2 + 1)}{x^2 + 1}$. (4)

(b) A curve C has the equation (6)

$$y = \sqrt{4x+1},\ x > -\tfrac{1}{4},\ y > 0.$$

The point P on the curve has x-coordinate 2. Find an equation of the tangent to C at P in the form $ax + by + c = 0$, where a, b, and c are integers.

18. The function g is defined by

$$\mathrm{g}(x) = \frac{\mathrm{e}^x - 3}{\mathrm{e}^x - 2},\ x \in \mathbb{R},\ x \neq \ln 2.$$

(a) Differentiate $\mathrm{g}(x)$ to show that $\mathrm{g}'(x) = \dfrac{\mathrm{e}^x}{(\mathrm{e}^x - 2)^2}$. (3)

(b) Find the exact values of x for which $\mathrm{g}'(x) = 1$. (4)

19. (a) Given that $y = \dfrac{\ln(x^2+1)}{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (4)

(b) Given that $x = \tan y$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1+x^2}$. (5)

20. (a) By writing $\sec x$ as $\dfrac{1}{\cos x}$, show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$. (3)

Given that $y = \mathrm{e}^{2x} \sec 3x$,

(b) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (4)

The curve with equation $y = \mathrm{e}^{2x} \sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b).

(c) Find the values of the constants a and b, giving your answers to 3 significant figures. (4)

21. A curve C has equation (7)

$$y = \frac{3}{(5-3x)^2},\ x \neq \tfrac{5}{3}.$$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b, and c are integers.

22. Figure 1 shows a sketch of the curve with the equation $y = (2x^2 - 5x + 2)\mathrm{e}^{-x}$.

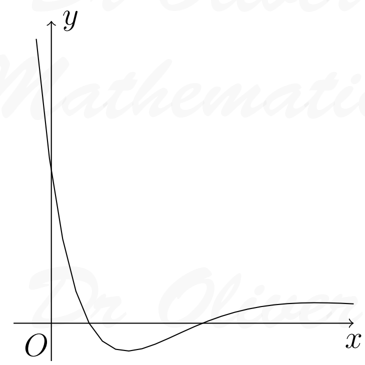

Figure 1: $y = (2x^2 - 5x + 2)e^{-x}$

(a) Find the coordinates of the point C where C crosses the y-axis. (1)

(b) Show that C crosses the x-axis at $x = 2$ and find the x-coordinate of the other point where C crosses the x-axis. (3)

(c) Find $\frac{dy}{dx}$. (3)

(d) Hence find the exact coordinates of the turning points of C. (5)

23. Given that

$$f(x) = \frac{4x - 1}{2(x - 1)} - \frac{3}{2(x - 1)(2x - 1)} - 2, \; x > 1,$$

(a) show that (6)

$$f(x) = \frac{3}{2x - 1}.$$

(b) Hence differentiate $f(x)$ and find $f'(2)$. (3)

24. (3)

$$f(x) = (8 - x) \ln x, \; x > 0.$$

Find $f'(x)$.

25. The curve C has equation

$$y = \frac{3 + \sin 2x}{2 + \cos 2x}.$$

(a) Show that (4)

$$\frac{dy}{dx} = \frac{6 \sin 2x + 4 \cos 2x + 2}{(2 + \cos 2x)^2}.$$

(b) Find an equation of the tangent to C at the point on C where $x = \frac{\pi}{2}$. (4)
Writhe your answer in the form $y = ax + b$, where a and b are exact constants.

26. (a) Given that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x,$$

show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$.

Given that

$$x = \sec 2y,$$

(b) find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y. (2)

(c) Hence find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (4)

27. Differentiate with respect to x:

(a) $\ln(x^2 + 3x + 5)$, (2)

(b) $\dfrac{\cos x}{x^2}$. (3)

28.

$$\mathrm{f}(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9},\ x \neq \pm 3,\ x \neq -\tfrac{1}{2}.$$

(a) Show that (5)

$$\mathrm{f}(x) = \frac{5}{(2x+1)(x+3)}.$$

The curve C has equation $y = \mathrm{f}(x)$. The point $P(-1, -\frac{5}{2})$ lies on C.

(b) Find an equation of the normal to C at P. (8)

29. Differentiate with respect to x, giving your answer in its simplest form:

(a) $x^2 \ln(3x)$, (4)

(b) $\dfrac{\sin 4x}{x^3}$. (5)

30. The point P is the point on the curve $x = 2\tan(y + \frac{\pi}{12})$ with y-coordinate $\frac{\pi}{4}$. (7)

Find an equation of the normal to the curve at P.

31. (4)

$$\mathrm{f}(x) = x^2 - 3x + 2\cos(\tfrac{1}{2}x),\ 0 \leqslant x \leqslant \pi.$$

The curve with equation $y = \mathrm{f}(x)$ has a minimum point P.

Show that the x-coordinate of P is the solution of the equation

$$x = \frac{3 + \sin(\frac{1}{2}x)}{2}.$$

32. Figure 2 shows a sketch of the curve C which has equation

$$y = e^{x\sqrt{3}} \sin 3x, \; -\tfrac{\pi}{3} \leqslant x \leqslant \tfrac{\pi}{3}.$$

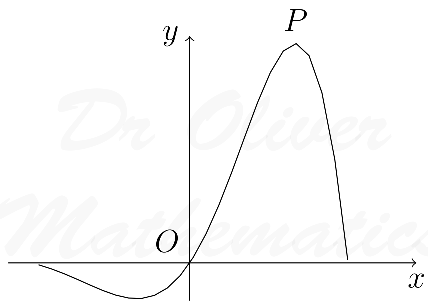

Figure 2: $y = e^{x\sqrt{3}} \sin 3x$

(a) Find the x-coordinate of the turning point P on C, for which $x > 0$. Give your answer as a multiple of π. (6)

(b) Find an equation of the normal to C at the point where $x = 0$. (3)

33. (a) Differentiate with respect to x:

(i) $x^{\frac{1}{2}} \ln(3x)$, (3)

(ii) $\dfrac{1 - 10x}{(2x-1)^5}$, giving your answer in its simplest form. (3)

(b) Given that $x = 3\tan 2y$, find $\dfrac{dy}{dx}$ in terms of x. (5)

34. The curve C has equation

$$y = (2x-3)^5.$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(a) the value of w, (2)

(b) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants. (5)

35. (a) Differentiate with respect to x:

(i) $y = x^3 \ln 2x$, (3)

(ii) $(x + \sin 2x)^3$. (3)

Given that $x = \cot y$,

(b) show that $\dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$. (5)

36.

$$f(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)},\ x \geqslant 0.$$

(a) Show that $f(x) = \dfrac{2x}{x^2+5}$. (4)

(b) Hence, or otherwise, find $f'(x)$ in its simplest form. (3)

Figure 3 shows a graph of the curve with equation $y = f(x)$.

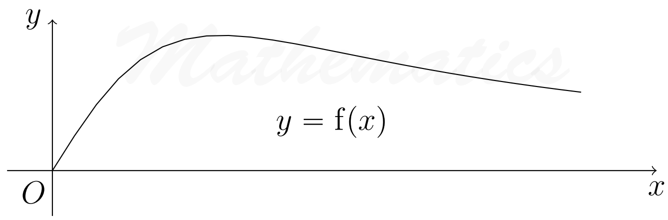

Figure 3: $f(x) = \dfrac{2}{x+2} - \dfrac{4}{x^2+5} - \dfrac{18}{(x^2+5)(x+2)}$

(c) Calculate the range of $f(x)$. (5)

37. (5)

$$f(x) = 25x^2e^{2x} - 16,\ x \in \mathbb{R}.$$

Using calculus, find the exact coordinates of the turning points on the curve with equation $y = f(x)$.

38. Given that

$$x = \sec^2 3y,\ 0 < y < \tfrac{\pi}{6},$$

(a) find $\dfrac{dx}{dy}$ in terms of y. (2)

(b) Hence show that (4)

$$\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}.$$

(c) Find an expression for $\dfrac{d^2y}{dx^2}$ in terms of x. Give your answer in its simplest form. (4)

39. (a) Differentiate (3)

$$\frac{\cos 2x}{\sqrt{x}}$$

with respect to x.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form (3)

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

(c) Given that $x = 2\sin\left(\frac{y}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, simplifying your answer. (4)

40. Figure 4 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}.$$

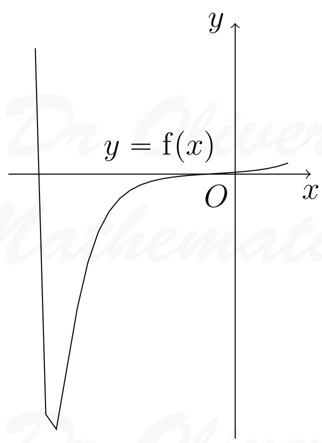

Figure 4: $\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}$

The curve cuts the x-axis at the points A and B.

(a) Calculate the x-coordinate of A and the x-coordinate of B, giving your answers to 3 decimal places. (2)

(b) Find $\mathrm{f}'(x)$. (3)

The curve has a minimum turning point at the point P.

(c) Show that the x-coordinate of P is the solution of (3)

$$x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)}.$$

41. The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x + 1}{x - 2},\ x > 2.$$

(a) Show that (3)

$$f'(x) = -\frac{9}{(x-2)^2}.$$

Given that P is a point on C such that $f'(x) = -1$,

(b) find the coordinates of P. (3)

42. The curve C has equation $x = 8y \tan 2y$.

The point P has coordinates $(\pi, \frac{\pi}{8})$.

(a) Verify that P lies on C. (1)

(b) Find the equation of the tangent to C at P in the form $ay = x + b$, where the constants a and b are to be found in terms of π. (7)

43. The curve C has equation (4)

$$y = 2\cos(\tfrac{1}{2}x^2) + x^3 - 3x - 2$$

and has a minimum turning point P.

Show that the x-coordinate of P is a solution of the equation

$$x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}.$$

44. A curve C has equation $y = e^{4x} + x^4 + 8x + 5$.

(a) Show that the x-coordinate of any turning point of C satisfies the equation (3)

$$x^3 = -2 - e^{4x}.$$

(b) Sketch, on a single diagram, the curves with equation (4)

(i) $y = x^3$,

(ii) $y = -2 - e^{4x}$. (2)

On your diagram give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

(c) Explain how your diagram illustrates that the equation $x^3 = -2 - e^{4x}$ has only one root. (1)

45. (a) Given that (4)

$$x = \sec 2y,\ 0 < y < \tfrac{\pi}{4},$$

show that

$$\frac{dy}{dx} = \frac{1}{4x\sqrt{x-1}}.$$

(b) Given that (5)

$$y = (x^2 + x^3)\ln 2x,$$

find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{\mathrm{e}}{2}$, giving your answer in its simplest form.

(c) Given that (3)

$$\mathrm{f}(x) = \frac{3\cos x}{(x+1)^{\frac{1}{3}}},\ x \neq -1,$$

show that

$$\mathrm{f}'(x) = \frac{\mathrm{g}(x)}{(x+1)^{\frac{4}{3}}},\ x \neq -1,$$

where $\mathrm{g}(x)$ is an expression to be found.

46. The point P lies on the curve with equation

$$x = (4y - \sin 2y)^2.$$

Given that P has (x, y) coordinates $(p, \frac{\pi}{2})$, where p is a constant,

(a) find the exact value of p. (1)

The tangent to the curve at P cuts the y-axis at the point A.

(b) Use calculus to find the coordinates of A. (6)

47. Figure 6 shows a sketch of part of the curve with equation

$$\mathrm{g}(x) = x^2(1-x)\mathrm{e}^{-2x},\ x \leqslant 0.$$

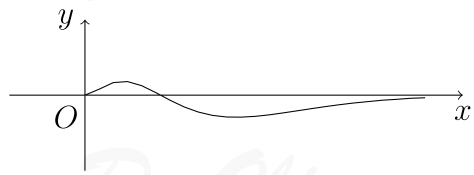

Figure 6: $\mathrm{g}(x) = x^2(1-x)\mathrm{e}^{-2x}$

(a) Show that $\mathrm{g}'(x) = \mathrm{f}(x)\mathrm{e}^{-2x}$, where $\mathrm{f}(x)$ is a cubic function to be found. (3)

(b) Hence find the range of g. (6)

(c) State a reason why the function $\mathrm{g}^{-1}(x)$ does not exist. (1)

48. Given that k is a **negative** constant and that the function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2},\ x \geqslant 0,$$

(a) show that $\mathrm{f}(x) = \dfrac{x+k}{x-2k}$. (3)

(b) Hence find $\mathrm{f}'(x)$, giving your answer in its simplest form. (3)

(c) State, with a reason, whether $\mathrm{f}(x)$ is an increasing or decreasing function. Justify your answer. (2)

49.

$$y = \frac{4x}{x^2+5}.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, writing your answer as a single fraction in its simplest form. (4)

(b) Hence find the set of values of x for which $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$. (3)

50. (a) Find, using calculus, the x-coordinate of the turning point of the curve with equation (5)

$$y = \mathrm{e}^{3x}\cos 4x,\ \tfrac{\pi}{4} \leqslant x < \tfrac{\pi}{2}.$$

Give your answer to 4 decimal places.

(b) Given $x = \sin^2 2y$, $0 < y < \frac{\pi}{4}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of y. (5)

Write your answer in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p\,\mathrm{cosec}(qy),$$

where p and q are constants to be determined.

51.

$$\mathrm{f}(x) = \frac{x^4+x^3-3x^2+7x-6}{x^2+x-6},\ x > 2,\ x \in \mathbb{R}.$$

(a) Given that (4)

$$\frac{x^4+x^3-3x^2+7x-6}{x^2+x-6} \equiv x^2 + A + \frac{b}{x-2},$$

find the values of the constants A and B.

(b) Hence, or otherwise, using calculus, find an equation of the normal to the curve with equation $y = \mathrm{f}(x)$ at the point where $x = 3$. (5)

52. Figure 7 shows a sketch of part of the curve C with equation

$$y = 2\ln(2x+5) - \tfrac{3}{2}x,\ x > -2.5.$$

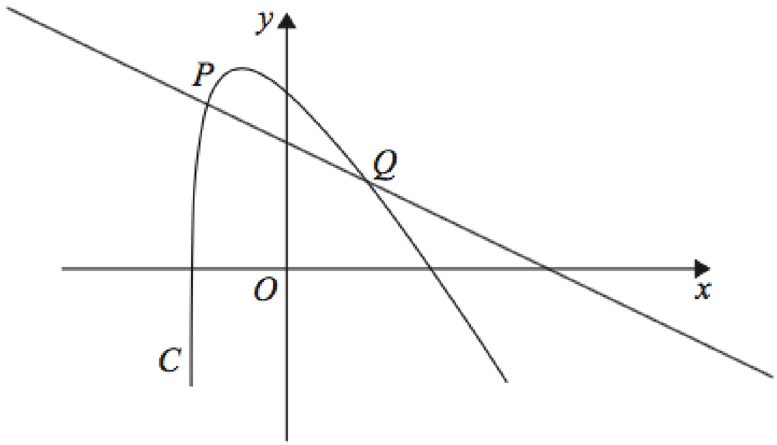

Figure 7: $y = 2\ln(2x+5) - \frac{3}{2}x$

The point P with x-coordinate -2 lies on C. Find an equation of the normal to C at P. Write your answer in the form $ax + by = c$, where a, b, and c are integers.

53. (a) Given $y = 2x(x^2-1)^5$, show that

(i) $\dfrac{dy}{dx} = g(x)(x^2-1)^4$, where $g(x)$ is a function to be determined. (4)

(ii) Hence find the set of values of x for which $\dfrac{dy}{dx} \geqslant 0$. (2)

(b) Given (4)

$$x = \ln(\sec 2y),\ 0 < y < \tfrac{\pi}{4},$$

find $\dfrac{dy}{dx}$ as a function of x in its simplest form.