Dr Oliver Mathematics

de Moivre's Theorem

Further Pure Mathematics 2, Chapter 3

Tangent

We *could* take expressions for sine and cosine and then divide them in order to get an appropriate expression for tangent. For example,

$$\sin 3\theta \equiv 3 \sin \theta - 4 \sin^3 \theta$$

and

$$\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta.$$

Hence

$$\tan 3\theta \equiv \frac{3 \sin \theta - 4 \sin^3 \theta}{4 \cos^3 \theta - 3 \cos \theta}$$

and all we need to do now is simplify in order to get an expression that involves only the tangent function.

An Expression for $\tan 3\theta$

$$\begin{aligned}
\tan 3\theta &\equiv \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \\
&\equiv \frac{\sin\theta\left[3 - 4\sin^2\theta\right]}{\cos\theta\left[4\cos^2\theta - 3\right]} \\
&\equiv \frac{\tan\theta\left[3(\sin^2\theta + \cos^2\theta) - 4\sin^2\theta\right]}{\left[4\cos^2\theta - 3(\sin^2\theta + \cos^2\theta)\right]} \\
&\equiv \frac{\tan\theta\left[3\cos^2\theta - \sin^2\theta\right]}{\cos^2\theta - 3\sin^2\theta} \\
&\equiv \frac{\tan\theta\left[3 - \tan^2\theta\right]}{1 - 3\tan^2\theta} \\
&\equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.
\end{aligned}$$

An Alternative Approach for Tangent

$$\begin{aligned}
[1 + i\tan\theta]^n &\equiv \left[1 + i\frac{\sin\theta}{\cos\theta}\right]^n \\
&\equiv \left[\frac{1}{\cos\theta}(\cos\theta + i\sin\theta)\right]^n \\
&\equiv \frac{1}{\cos^n\theta}(\cos\theta + i\sin\theta)^n \\
&\equiv \frac{1}{\cos^n\theta}(\cos n\theta + i\sin n\theta).
\end{aligned}$$

Hence

$$\operatorname{Re}[1 + i\tan\theta]^n \equiv \frac{\cos n\theta}{\cos^n\theta} \text{ and } \operatorname{Im}[1 + i\tan\theta]^n \equiv \frac{\sin n\theta}{\cos^n\theta}$$

so

$$\tan n\theta \equiv \frac{\operatorname{Im}[1 + i\tan\theta]^n}{\operatorname{Re}[1 + i\tan\theta]^n}.$$

An Expression for $\tan 3\theta$

$$\begin{aligned}[1 + i\tan\theta]^3 &\equiv 1 + 3(i\tan\theta) + 3(i\tan\theta)^2 + (i\tan\theta)^3 \\ &\equiv \left(1 - 3\tan^2\theta\right) + i\left(3\tan\theta - \tan^3\theta\right).\end{aligned}$$

So

$$\operatorname{Im}[1 + i\tan\theta]^3 \equiv 3\tan\theta - \tan^3\theta$$

and

$$\operatorname{Re}[1 + i\tan\theta]^3 \equiv 1 - 3\tan^2\theta.$$

Hence

$$\tan 3\theta \equiv \frac{\operatorname{Im}[1 + i\tan\theta]^3}{\operatorname{Re}[1 + i\tan\theta]^3} \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

Let

$$z = \cos\theta + i\sin\theta.$$

Then

$$\begin{aligned} z^n &= (\cos\theta + i\sin\theta)^n \\ &= \cos n\theta + i\sin n\theta \end{aligned}$$

and

$$\begin{aligned} z^{-n} &= (\cos\theta + i\sin\theta)^{-n} \\ &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta, \end{aligned}$$

since sine and cosine are odd and even functions respectively.

From powers to multiple angles

$$\begin{aligned} z^n + \frac{1}{z^n} &= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \\ &= 2 \cos n\theta \end{aligned}$$

and so

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

In the same way,

$$\begin{aligned} z^n - \frac{1}{z^n} &= [\cos n\theta + i \sin n\theta] + [\cos n\theta - i \sin n\theta] \\ &= 2 \sin n\theta \end{aligned}$$

and so

$$\sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right).$$

Writing $\cos 3\theta$ using multiple angles

$$\begin{aligned}
\cos^3\theta &= [\cos\theta]^3 \\
&= \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^3 \\
&= \frac{1}{8}\left[z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right] \\
&= \frac{1}{4}\left[\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)+3\times\frac{1}{2}\left(z+\frac{1}{z}\right)\right] \\
&= \frac{1}{4}\left[\cos 3\theta+3\cos\theta\right] \\
&= \frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta.
\end{aligned}$$

Writing $\sin 3\theta$ using multiple angles

$$\begin{aligned}
\sin^3\theta &= [\sin\theta]^3 \\
&= \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^3 \\
&= -\frac{1}{8i}\left[z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}\right] \\
&= -\frac{1}{4}\left[\frac{1}{2i}\left(z^3 - \frac{1}{z^3}\right) - 3 \times \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \\
&= -\frac{1}{4}\left[\sin 3\theta - 3\sin\theta\right] \\
&= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta.
\end{aligned}$$