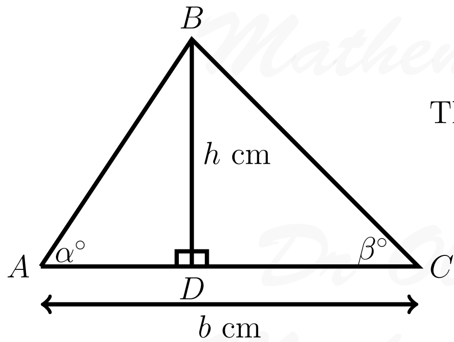
Dr Oliver Mathematics
Two Formulae for the Height of a Triangle

In this note, we present two methods for finding the height of a triangle.

1 Acute Triangle

Figure 1 shows the basic set-up:



The diagram is **NOT** to scale

Figure 1: the basic set-up

We have

- a triangle ABC,
- $b = AC$ is the base,
- $h = BD$ is the height,
- $\alpha^\circ = \angle BAC$, and
- $\beta^\circ = \angle BCA$.

We split b up in two parts: $b_1 = AD$ and $b_2 = DC$, as seen in Figure 2:

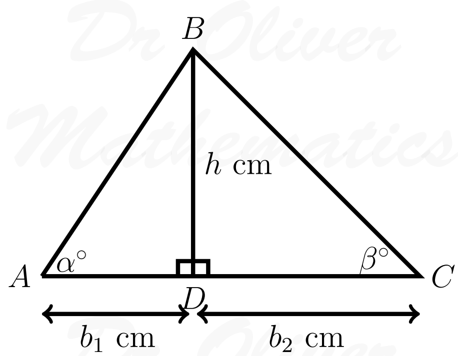


Figure 2: b_1 and b_2

Now,

$$\begin{aligned}\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan \alpha^\circ &= \frac{h}{b_1} \\ \Rightarrow b_1 &= \frac{h}{\tan \alpha^\circ}\end{aligned}$$

and

$$\begin{aligned}\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan \beta^\circ &= \frac{h}{b_2} \\ \Rightarrow b_2 &= \frac{h}{\tan \beta^\circ}.\end{aligned}$$

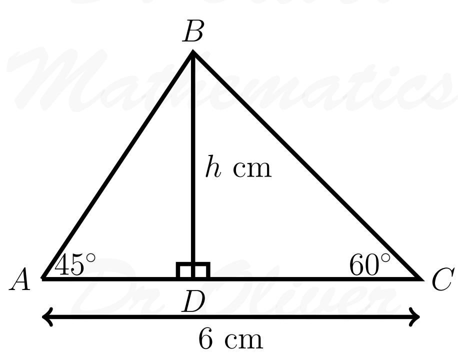
Next,

$$\begin{aligned}b &= b_1 + b_2 \\ &= \frac{h}{\tan \alpha^\circ} + \frac{h}{\tan \beta^\circ} \\ &= h\left(\frac{1}{\tan \alpha^\circ} + \frac{1}{\tan \beta^\circ}\right) \\ &= h\left(\frac{\tan \beta^\circ}{\tan \alpha^\circ \tan \beta^\circ} + \frac{\tan \alpha^\circ}{\tan \alpha^\circ \tan \beta^\circ}\right) \\ &= h\left(\frac{\tan \alpha^\circ + \tan \beta^\circ}{\tan \alpha^\circ \tan \beta^\circ}\right).\end{aligned}$$

Finally,

$$b = h\left(\frac{\tan \alpha^\circ + \tan \beta^\circ}{\tan \alpha^\circ \tan \beta^\circ}\right) \Rightarrow \boxed{h = \frac{b \tan \alpha^\circ \tan \beta^\circ}{\tan \alpha^\circ + \tan \beta^\circ}.}$$

Example 1
We have a triangle ABC.



Find h.

Solution 1
Now,

$$\begin{aligned} h &= \frac{6 \tan 45^\circ \tan 60^\circ}{\tan 45^\circ + \tan 60^\circ} \\ &= \frac{6(1)(\sqrt{3})}{1 + \sqrt{3}} \\ &= \frac{6\sqrt{3}}{1 + \sqrt{3}} \\ &= \left(\frac{6\sqrt{3}}{1 + \sqrt{3}}\right) \times \left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right) \end{aligned}$$

$\times$	1	$+\sqrt{3}$
1	1	$+\sqrt{3}$
$-\sqrt{3}$	$-\sqrt{3}$	-3

$$\begin{aligned} &= \frac{6\sqrt{3} - 18}{1 - 3} \\ &= \frac{-2(9 - 3\sqrt{3})}{-2} \\ &= \underline{\underline{(9 - 3\sqrt{3}) \text{ cm}}}. \end{aligned}$$

2 Obtuse Triangle

Suppose you a have a tower and you do not know how high it is. How do find out? Well, one way is

- make a mark in the ground and record the angle of elevation (α°, say), up to the top of the tower,
- go along towards the tower,
- make a mark in the ground and record the angle of elevation (β°, say), up to the top of the tower, and
- use trigonometry to work it out.

I used to, when working with Year 9, take them in to the hall (or, outside if the weather permits) and measure the height of hall (or, outside, the top of the PE block).

Figure 3 shows the basic set-up:

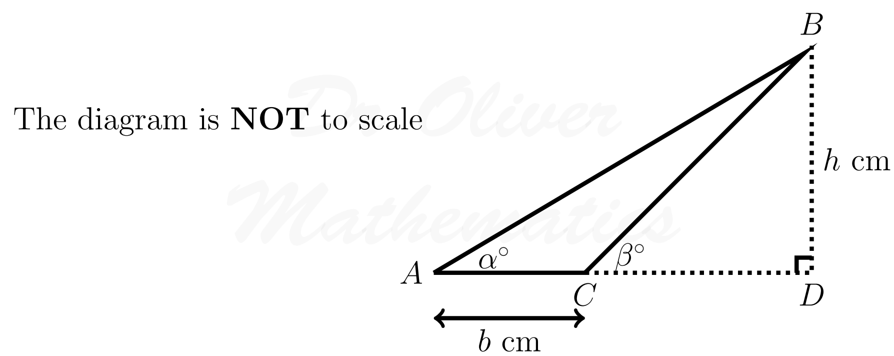


Figure 3: the basic set-up

We have

- a triangle ABC,
- $b = AC$ is the base,
- $h = BD$ is the height,
- $\alpha^\circ = \angle BAC$, and
- $\beta^\circ = \angle BCD$.

We split b up in two parts: $b_1 = AD$ and $b_2 = CD$, as seen in Figure 4:

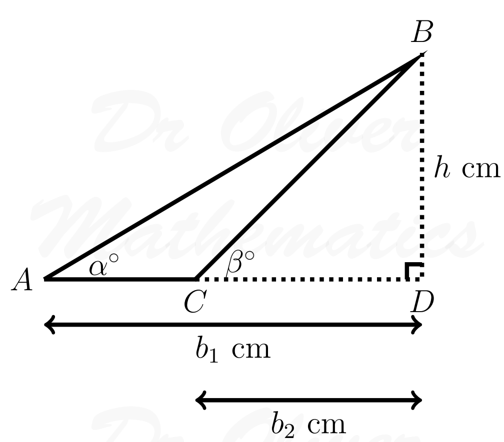


Figure 4: b_1 and b_2

Now,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\alpha^\circ = \frac{h}{b_1}$$
$$\Rightarrow b_1 = \frac{h}{\tan\alpha^\circ}$$

and

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\beta^\circ = \frac{h}{b_2}$$
$$\Rightarrow b_2 = \frac{h}{\tan\beta^\circ}.$$

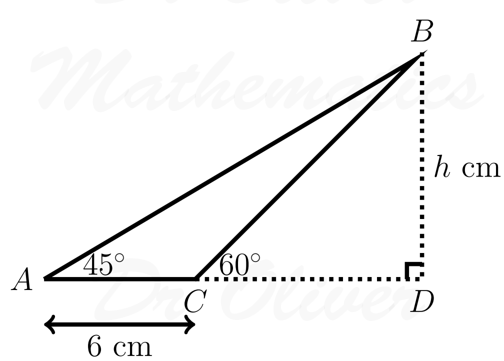
Next,

$$\begin{aligned} b &= b_1 - b_2 \\ &= \frac{h}{\tan\alpha^\circ} - \frac{h}{\tan\beta^\circ} \\ &= h\left(\frac{1}{\tan\alpha^\circ} - \frac{1}{\tan\beta^\circ}\right) \\ &= h\left(\frac{\tan\beta^\circ}{\tan\alpha^\circ\tan\beta^\circ} - \frac{\tan\alpha^\circ}{\tan\alpha^\circ\tan\beta^\circ}\right) \\ &= h\left(\frac{\tan\beta^\circ - \tan\alpha^\circ}{\tan\alpha^\circ\tan\beta^\circ}\right). \end{aligned}$$

Finally,

$$b = h\left(\frac{\tan\beta^\circ - \tan\alpha^\circ}{\tan\alpha^\circ\tan\beta^\circ}\right) \Rightarrow \boxed{h = \frac{b\tan\alpha^\circ\tan\beta^\circ}{\tan\beta^\circ - \tan\alpha^\circ}.}$$

Example 2
We have a triangle ABC.



Find h.

Solution 2
Now,

$$\begin{aligned} h &= \frac{6 \tan 45^\circ \tan 60^\circ}{\tan 60^\circ - \tan 45^\circ} \\ &= \frac{6(1)(\sqrt{3})}{\sqrt{3} - 1} \\ &= \frac{6\sqrt{3}}{\sqrt{3} - 1} \\ &= \left(\frac{6\sqrt{3}}{\sqrt{3} - 1} \right) \times \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1} \right) \end{aligned}$$

$\times$	$\sqrt{3}$	$+1$
$\sqrt{3}$	3	$+\sqrt{3}$
-1	$-\sqrt{3}$	-1

$$\begin{aligned} &= \frac{18 + 6\sqrt{3}}{3 - 1} \\ &= \frac{2(9 + 3\sqrt{3})}{2} \\ &= \underline{\underline{(9 + 3\sqrt{3}) \text{ cm}}}. \end{aligned}$$