Dr Oliver Mathematics
AQA Further Maths Level 2
June 2015 Paper 1
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.
You are **not** permitted to use a scientific or graphical calculator in this paper.

1. GH is a straight line. (2)

   The coordinates of G are $(-2, 8)$.
   The midpoint of GH is $(5, -3)$.

   Work out the coordinates of H.

Solution

Let $H(x, y)$. Then

$$(5, -3) = \left(\frac{-2 + x}{2}, \frac{8 + y}{2}\right).$$

Now,

$$\begin{aligned}\frac{-2 + x}{2} = 5 &\Rightarrow -2 + x = 10 \\ &\Rightarrow x = 12\end{aligned}$$

and

$$\begin{aligned}\frac{8 + y}{2} = -3 &\Rightarrow 8 + y = -6 \\ &\Rightarrow y = -14.\end{aligned}$$

Hence, $\underline{\underline{H(12, -14)}}$.

2. A straight line with equation $y = mx + c$ has gradient m and y-intercept c.
   Here are the equations of four straight lines: P, Q, R, and S.

$$\begin{aligned} P: &\quad 2y - 4x = 5 \\ Q: &\quad 5y = 2x - 4 \\ R: &\quad 2y - 4 = 5x \\ S: &\quad 4y = 5 - 2x. \end{aligned}$$

(a) Circle the line that passes through $(7, 2)$. (1)

$P \quad Q \quad R \quad S$

Solution

$\underline{\underline{Q}}$.

(b) Circle the line with gradient $2\frac{1}{2}$. (1)

$P \quad Q \quad R \quad S$

Solution

$\underline{\underline{R}}$.

(c) Circle the line with y-intercept $2\frac{1}{2}$. (1)

$P \quad Q \quad R \quad S$

Solution

$\underline{\underline{P}}$.

(d) Circle the line with a negative gradient. (1)

$P \quad Q \quad R \quad S$

Solution

$\underline{\underline{S}}$.

(e) Circle a pair of perpendicular lines. (1)

$P \quad Q \quad R \quad S$

Solution

$\underline{\underline{P \text{ and } S}}$.

3. Solve (2)

$$2(3x+1) > 3 - 4x.$$

Solution

$$\begin{aligned} 2(3x+1) > 3 - 4x &\Rightarrow 6x + 2 > 3 - 4x \\ &\Rightarrow 10x > 1 \\ &\Rightarrow \underline{\underline{x > \tfrac{1}{10}}}. \end{aligned}$$

4. The equation of a curve is

$$y = x^2 - 5x.$$

(a) Work out $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (2)

Solution

$$y = x^2 - 5x \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 5}}.$$

P is a point on the curve.
The tangent to the curve at P has gradient 1.

(b) Work out the coordinates of P. (2)

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 1 &\Rightarrow 2x - 5 = 1 \\ &\Rightarrow 2x = 6 \\ &\Rightarrow x = 3 \end{aligned}$$

and

$$y = 3^2 - 5(3) = 9 - 15 = -6.$$

Hence, $\underline{\underline{P(3, -6)}}$.

5. In the expansion of

$$(x+2)(x^2+kx-3)$$

the coefficient of x^2 is zero.

(a) Work out the value of k. (1)

Solution

$\times$	x^2	$+kx$	-3
x	x^3	$+kx^2$	$-3x$
$+2$	$+2x^2$	$+2kx$	-6

So

$$(x+2)(x^2+kx-3) = x^3 + (k+2)x^2 + (2k-3)x - 6.$$

Hence, $\underline{\underline{k = -2}}$.

(b) Work out the value of x. (2)

Solution

$$2(-3) - 1 = \underline{\underline{-7}}.$$

6. A bag contains $5x$ red balls and $2x$ blue balls. (4)

The number of red balls is **decreased** by 20%.
The number of blue balls is **increased** by 30%.
There are now 35 more red balls than blue balls in the bag.

Work out the value of x.

Solution

After the adjustment, the bag contains

$$\begin{aligned}(1-0.2)(5x) &= (0.8)(5x)\\ &= 4x\end{aligned}$$

red balls and

$$\begin{aligned}(1+0.3)(2x) &= (1.3)(2x)\\ &= 2.6x\end{aligned}$$

blue balls. Now,

$$\begin{aligned}4x - 2.6x = 35 &\Rightarrow 1.4x = 35\\ &\Rightarrow x = \frac{35}{1.4}\\ &\Rightarrow x = 35 \times \frac{5}{7}\\ &\Rightarrow x = 5 \times 5\\ &\Rightarrow \underline{\underline{x = 25}}.\end{aligned}$$

7. (3)

$$3x^3 - 2x^2 - 147x + 98 \equiv (ax - c)(bx + d)(bx - d)$$

where a, b, c, and d are positive integers.

Work out the values of a, b, c, and d.

Solution

$$\begin{array}{r|l} & 98 \\ \hline 2 & 49 \\ \hline 7 & 7 \\ \hline 7 & 1 \\ \hline \end{array}$$

So $\underline{\underline{c = 2}}$ and $\underline{\underline{d = 7}}$. Now,

$$3 = ab^2 \Rightarrow \underline{\underline{a = 3}} \text{ and } \underline{\underline{b = 1}}.$$

8. Simplify fully (4)

$$\frac{5x}{(x+4)(x-6)} - \frac{3}{x-6}.$$

Solution

$$\begin{aligned}\frac{5x}{(x+4)(x-6)} - \frac{3}{x-6} &= \frac{5x}{(x+4)(x-6)} - \frac{3(x-4)}{(x+4)(x-6)} \\ &= \frac{5x-3(x+4)}{(x+4)(x-6)} \\ &= \frac{5x-3x-12)}{(x+4)(x-6)} \\ &= \frac{2x-12)}{(x+4)(x-6)} \\ &= \frac{2(x-6)}{(x+4)(x-6)} \\ &= \underline{\underline{\frac{2}{x+4}}}.\end{aligned}$$

9. Given that (5)

$$\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a+1 \end{pmatrix},$$

work out the values of a and b.

Solution

$$\begin{aligned} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} b \\ a+1 \end{pmatrix} \\ \Rightarrow \quad \begin{pmatrix} 3a-b \\ 2a+b \end{pmatrix} &= \begin{pmatrix} b \\ a+1 \end{pmatrix} \end{aligned}$$

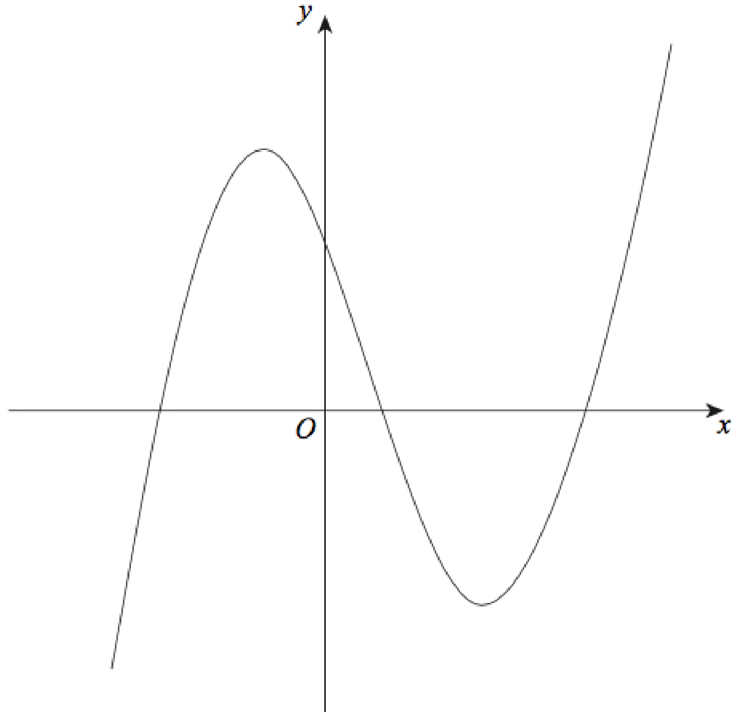
which means

$$3a - b = b \Rightarrow 3a = 2b \quad (1)$$
$$2a + b = a + 1 \Rightarrow b = 1 - a \quad (2).$$

Substitute (2) into (1):

$$\begin{aligned} 3a = 2(1-a) &\Rightarrow 3a = 2 - 2a \\ &\Rightarrow 5a = 2 \\ &\Rightarrow \underline{\underline{a = \tfrac{2}{5}}} \\ &\Rightarrow \underline{\underline{b = \tfrac{3}{5}}}. \end{aligned}$$

10. This is a sketch of the curve $y = \mathrm{f}(x)$.



For this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x - 4.$$

(a) Work out the range of values of x for which $\mathrm{f}(x)$ is a decreasing function. Write your answer as an inequality. (4)

Solution

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & (+3) \times (-4) = -12 \end{array}\right\} -6,\ +2$$

Now,

$$\begin{aligned} 3x^2 - 4x - 4 = 0 &\Rightarrow 3x^2 - 6x + 2x - 4 = 0 \\ &\Rightarrow 3x(x-2) + 2(x-2) = 0 \\ &\Rightarrow (3x+2)(x-2) = 0 \\ &\Rightarrow x = -\tfrac{2}{3} \text{ or } x = 2. \end{aligned}$$

Finally, $\mathrm{f}(x)$ is a decreasing function when

$$\underline{\underline{-\tfrac{2}{3} < x < 2.}}$$

(b) Work out the equation of the normal to the curve at the point $(1, -2)$. (5)
Give your answer in the form $y = mx + c$.

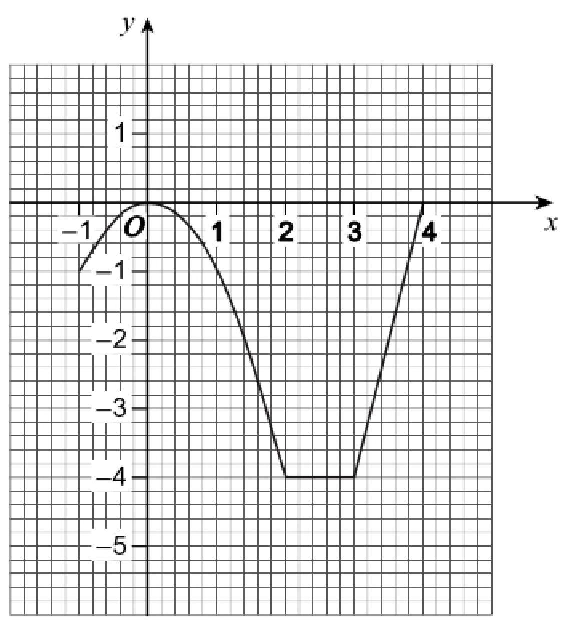
Solution

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 4 - 4$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -5$$
$$\Rightarrow m_{\text{normal}} = \tfrac{1}{5}.$$

Hence, the equation of the normal is

$$y + 2 = \tfrac{1}{5}(x - 1) \Rightarrow y + 2 = \tfrac{1}{5}x - \tfrac{1}{5}$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{5}x - \tfrac{11}{5}}}.$$

11. Here is the graph of $y = \mathrm{f}(x)$. (4)
It consists of a quadratic curve and two straight lines.



Define $\mathrm{f}(x)$, stating clearly the domain for each part.

Solution

$$\mathrm{f}(x) = \begin{cases} \underline{\underline{-x^2}}, & \underline{\underline{-1 \leqslant x \leqslant 2}}, \\ \underline{\underline{-4}}, & \underline{\underline{2 < x < 3}} \\ \underline{\underline{4x - 16}}, & \underline{\underline{3 \leqslant x \leqslant 4}}. \end{cases}$$

12. Make y the subject for (4)

$$\sqrt{\frac{3xy}{x+y}} = 4.$$

Solution

$$\begin{aligned} \sqrt{\frac{3xy}{x+y}} = 4 &\Rightarrow \frac{3xy}{x+y} = 16 \\ &\Rightarrow 3xy = 16(x+y) \\ &\Rightarrow 3xy = 16x + 16y \\ &\Rightarrow 3xy - 16y = 16x \\ &\Rightarrow y(3x - 16) = 16x \\ &\Rightarrow \underline{\underline{y = \frac{16x}{3x - 16}}}. \end{aligned}$$

13. (3)

$$x^2 + 2ax + b \equiv (x-5)^2 - a.$$

Work out the values of a and b.

Solution

$\times$	x	-5
x	x^2	$-5x$
-5	$-5x$	$+25$

So,

$$x^2 + 2ax + b \equiv x^2 - 10x + (25 - a).$$

Well,

$$2a = -10 \Rightarrow \underline{\underline{a = -5}}$$
$$\Rightarrow b = 25 - (-5) = \underline{\underline{30}}.$$

14. Write (5)

$$\frac{5\sqrt{2}}{3\sqrt{6} - 7}$$

in the form $\sqrt{w} + \sqrt{k}$ where w and k are integers.

Solution

$\times$	$3\sqrt{6}$	-7
$3\sqrt{6}$	54	$-21\sqrt{6}$
$+7$	$+21\sqrt{6}$	-49

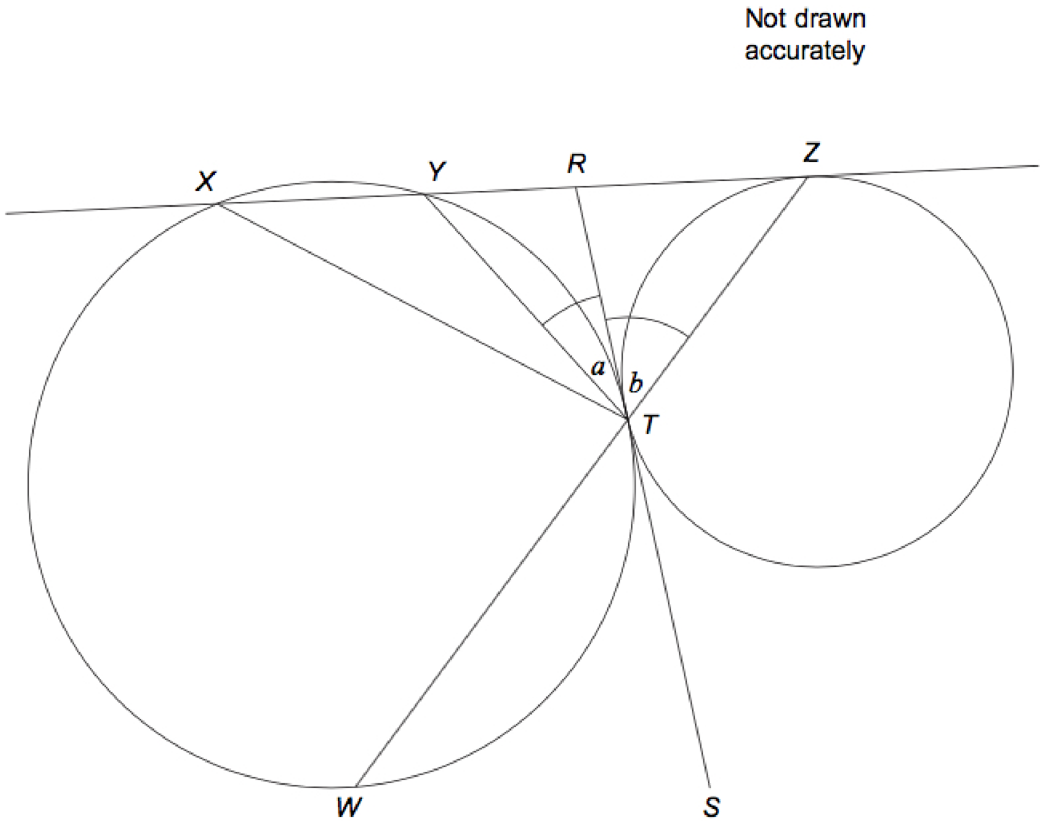
$$\begin{aligned}
\frac{5\sqrt{2}}{3\sqrt{6} - 7} &= \frac{5\sqrt{2}}{3\sqrt{6} - 7} \times \frac{3\sqrt{6} + 7}{3\sqrt{6} + 7} \\
&= \frac{5\sqrt{2}(3\sqrt{6} + 7)}{54 - 49} \\
&= \frac{15\sqrt{12} + 35\sqrt{2}}{5} \\
&= 3\sqrt{12} + 7\sqrt{2} \\
&= (\sqrt{9} \times \sqrt{12}) + (\sqrt{49} \times \sqrt{2}) \\
&= \sqrt{9 \times 12} + \sqrt{49 \times 2} \\
&= \underline{\underline{\sqrt{108} + \sqrt{98}}}.
\end{aligned}$$

15. The diagram shows two circles touching externally at T.
Points X, Y, and W lie on the larger circle.

- RTS is a tangent to both circles.

- $XYRZ$ is a tangent to the smaller circle at Z.
- ZTW is a straight line.

Angle $YTR = a$ and angle $ZTR = b$.



(a) Give reasons why angle $RZT = b$. (2)

Solution
Two tangents drawn from an external point to a circle are the same length and that means the triangle $\triangle RTZ$ is isosceles and $\angle RZT = b$.

Angle $RZT = b$.

(b) Prove that (3)

$$\text{angle } XTW = \text{angle } YTZ.$$

Solution

$\angle YTZ = \angle YTR + \angle RTZ = a + b.$

Now, $\angle RTZ = 180 - 2b$ (completing the triangle)
$\angle YRT = 2b$ (supplementary angles)
$\angle TYR = 180 - a - 2b = 180 - (a + 2b)$ (completing the triangle)
$\angle XYT = a + 2b$ (supplementary angles)
$\angle TXY = a$ (alternate segment theorem)
$\angle XTY = 180 - a - (a + 2b) = 180 - 2b$ (completing the triangle)
$\angle XTW = 180 - a - (180 - 2b) - b = a + b$ (supplementary angles)

Hence, $\underline{\underline{\angle XTW = \angle YTZ}}$, as required.

16. By factorising fully, simplify (5)

$$\frac{x^4 - x^3 - 2x^2}{x^4 - 5x^2 + 4}.$$

Solution

$$x^4 - x^3 - 2x^2 = x^2(x^2 - x - 2)$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -2 \end{array}\right\} -2,\ +1$$

$$= x^2(x - 2)(x + 1)$$

and

$$\left.\begin{array}{ll} \text{add to:} & -5 \\ \text{multiply to:} & +4 \end{array}\right\} -4,\ -1$$

$$\begin{aligned} x^4 - 5x^2 + 4 &= (x^2)^2 - 5(x^2) + 4 \\ &= (x^2 - 4)(x^2 - 1) \\ &= (x - 2)(x + 2)(x + 1)(x - 1), \end{aligned}$$

using the difference of two squares. Finally,

$$\begin{aligned} \frac{x^4 - x^3 - 2x^2}{x^4 - 5x^2 + 4} &= \frac{x^2(x - 2)(x + 1)}{(x - 2)(x + 2)(x + 1)(x - 1)} \\ &= \underline{\underline{\frac{x^2}{(x + 2)(x - 1)}}}. \end{aligned}$$

17. Prove that (3)

$$2\tan^2\theta + 1 \equiv \frac{1+\sin^2\theta}{1-\sin^2\theta},$$

where $\sin^2\theta \neq 1$.

Solution

$$\begin{aligned}
\frac{1+\sin^2\theta}{1-\sin^2\theta} &\equiv \frac{1+\sin^2\theta}{\cos^2\theta} \\
&\equiv \frac{(\sin^2\theta+\cos^2\theta)+\sin^2\theta}{\cos^2\theta} \\
&\equiv \frac{2\sin^2\theta+\cos^2\theta}{\cos^2\theta} \\
&\equiv \frac{2\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \\
&\equiv \underline{\underline{2\tan^2\theta + 1}},
\end{aligned}$$

as required.