Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 November Paper 2 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Solve the equation (3)

$$|2x + 10| = 7.$$

Solution

$\underline{2x + 10 = 7}$:

$$\begin{aligned} 2x + 10 = 7 &\Rightarrow 2x = -3 \\ &\Rightarrow x = -1\tfrac{1}{2}. \end{aligned}$$

$\underline{-(2x + 10) = 7}$:

$$\begin{aligned} -(2x + 10) = 7 &\Rightarrow -2x - 10 = 7 \\ &\Rightarrow -2x = 17 \\ &\Rightarrow x = -8\tfrac{1}{2}. \end{aligned}$$

Hence,

$$|2x + 10| = 7 \Rightarrow \underline{\underline{x = -8\tfrac{1}{2} \text{ or } x = -1\tfrac{1}{2}}}.$$

2. The expression (5)

$$x^3 + ax^2 - 15x + b$$

has a factor $(x - 2)$ and leaves a remainder of 75 when divided by $(x + 3)$.

Find the value of a and of b.

Solution

We use synthetic division twice:

$$\begin{array}{r|cccc} 2 & 1 & a & -15 & b \\ & \downarrow & 2 & 2a+4 & 4a-22 \\ \hline & 1 & a+2 & 2a-11 & 4a+b-22 \end{array}$$

and so

$$4a + b - 22 = 0 \Rightarrow 4a + b = 22 \quad (1).$$

$$\begin{array}{r|cccc} -3 & 1 & a & -15 & b \\ & \downarrow & -3 & -3a+9 & 9a+18 \\ \hline & 1 & a-3 & -3a-6 & 9a+b+18 \end{array}$$

and so

$$9a + b + 18 = 75 \Rightarrow 9a + b = 57 \quad (2).$$

Do (2) − (1):

$$5a = 35 \Rightarrow \underline{\underline{a = 7}}.$$

Insert this in to (1):

$$4(7) + b = 22 \Rightarrow \underline{\underline{b = -6}}.$$

[Check: $9(7) + (-6) = 57$ ✓]

3. A number, N_0, of fish of a particular species are introduced to a lake.

The number, N, of these fish in the lake, t weeks after their introduction, is given by

$$N = N_0 e^{-kt},$$

where k is a constant.

Calculate

(a) the value of k if, after 34 weeks, the number of these fish has fallen to $\frac{1}{2}$ of the number introduced, (2)

Solution

Now,

$$\begin{aligned} t = 34 \Rightarrow \tfrac{1}{2}N_0 = N_0\mathrm{e}^{-34k} \\ &\Rightarrow \mathrm{e}^{-34k} = \tfrac{1}{2} \\ &\Rightarrow \mathrm{e}^{34k} = 2 \\ &\Rightarrow 34k = \ln 2 \\ &\Rightarrow \underline{\underline{k = \tfrac{1}{34}\ln 2}}. \end{aligned}$$

(b) the number of weeks it takes for the number of these fish to have fallen to $\frac{1}{5}$ of the number introduced. (3)

Solution

Well,

$$\begin{aligned} \tfrac{1}{5}N_0 = N_0\mathrm{e}^{-(\frac{1}{34}\ln 2)t} &\Rightarrow \tfrac{1}{5} = \mathrm{e}^{-(\frac{1}{34}\ln 2)t} \\ &\Rightarrow \mathrm{e}^{(\frac{1}{34}\ln 2)t} = 5 \\ &\Rightarrow (\tfrac{1}{34}\ln 2)t = \ln 5 \\ &\Rightarrow t = \frac{34\ln 5}{\ln 2} \\ &\Rightarrow t = 78.945\,555\,23 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 79 \text{ weeks (nearest whole number)}}}. \end{aligned}$$

4. Students take three multiple-choice tests, each with ten questions. (5)

- A correct answer earns 5 marks.
- If no answer is given 1 mark is scored.
- An incorrect answer loses 2 marks.

A student's final total mark is the sum of

- 20% of the mark in test 1,
- 30% of the mark in test 2, and
- 50% of the mark in test 3.

One student's responses are summarized in the table below.

	Test 1	Test 2	Test 3
Correct answer	7	6	5
No answer	1	3	5
Incorrect answer	2	1	0

Write down three matrices such that matrix multiplication will give this student's final total mark and hence find this total mark.

Solution

Well,

$$\begin{aligned} &\underline{\underline{\begin{pmatrix} 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} 7 & 6 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.3 \\ 0.5 \end{pmatrix}}} \\ &= \begin{pmatrix} 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} 5.7 \\ 3.6 \\ 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 30.7 \end{pmatrix}; \end{aligned}$$

hence,

$$\underline{\underline{\text{total mark} = 30.7}}.$$

5. Find the set of values of m for which the line (6)

$$y = mx - 2$$

cuts the curve

$$y = x^2 + 8x + 7$$

in two distinct points.

Solution

Well,

$$x^2 + 8x + 7 = mx - 2 \Rightarrow x^2 + (8 - m)x + 9 = 0.$$

Now, $a = 1$, $b = 8 - m$, and $c = 9$:

$\times$	8	$-m$
8	64	$-8m$
$-m$	$-8m$	$+m^2$

and

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow (8-m)^2 - 4(1)(9) > 0 \\ &\Rightarrow (64 - 16m + m^2) - 36 > 0 \\ &\Rightarrow m^2 - 16m + 28 > 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -16 \\ \text{multiply to:} & +28 \end{array}\right\} -14,\ -2$$

$$\Rightarrow (m-14)(m-2) > 0.$$

We need a 'table of signs':

	$m<2$	$m=2$	$2<m<14$	$m=14$	$m>14$
$m-2$	$-$	0	$+$	$+$	$+$
$m-14$	$-$	$-$	$-$	0	$+$
$(m-2)(m-14)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{m < 2 \text{ or } m > 14}}.$$

6. A 4-digit number is formed by using four of the seven digits 1, 3, 4, 5, 7, 8, and 9. No digit can be used more than once in any one number.

   Find how many different 4-digit numbers can be formed if

   (a) there are no restrictions, (2)

   Solution

   $$7 \times 6 \times 5 \times 4 = \underline{\underline{840}}.$$

   (b) the number is less than 4 000, (2)

Solution

There are only two (1 and 3) for the first number:

$$2 \times 6 \times 5 \times 4 = \underline{\underline{240}}.$$

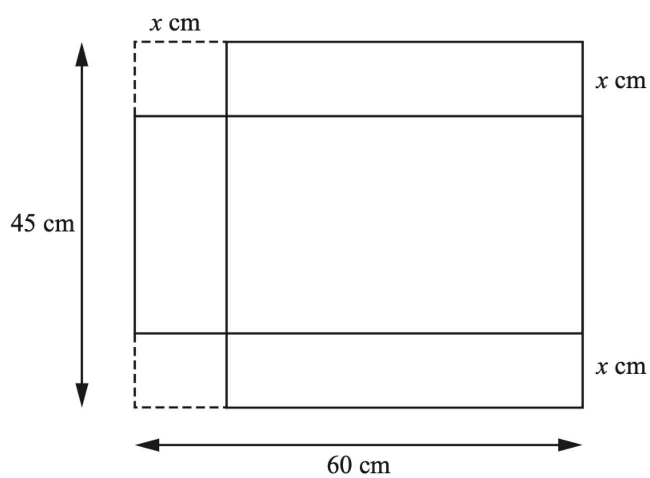
(c) the number is even and less than 4 000. (2)

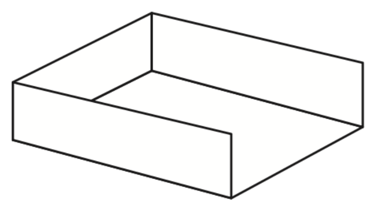
Solution

There are only two (4 and 8) for the last number:

$$2 \times 5 \times 4 \times 2 = \underline{\underline{80}}.$$

7. A rectangular sheet of metal measures 60 cm by 45 cm.





A scoop is made by cutting out squares, of side x cm, from two corners of the sheet and folding the remainder as shown.

(a) Show that the volume, V cm^3, of the scoop is given by (2)

$$V = 2\,700x - 165x^2 + 2x^3.$$

Solution

Well,

$$V = x(45 - 2x)(60 - x)$$

×	45	$-2x$
60	2 700	$-120x$
$-x$	$-45x$	$+2x^2$

$$= x(2\,700 - 165x + 2x^2)$$
$$= \underline{\underline{2\,700x - 165x^2 + 2x^3}},$$

as required.

(b) Given that x can vary, find the value of x for which V has a stationary value. (4)

Solution

Now,

$$V = 2\,700x - 165x^2 + 2x^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}x} = 2\,700 - 330x + 6x^2$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow 6x^2 - 330x + 2\,700 = 0$$
$$\Rightarrow 6(x^2 - 55x - 450) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -55 \\ \text{multiply to:} & +450\end{array}\right\} -45,\ -10$$

$$\Rightarrow 6(x - 45)(x - 10) = 0$$
$$\Rightarrow x = 45 \text{ or } x = 10.$$

But $x \neq 45$ (why?) and, hence, $\underline{\underline{x = 10}}$.

8. Solve the equation

(a) (4)

$$\log_{10}(5x + 10) + 2\log_{10} 3 = 1 + \log_{10}(4x + 12),$$

Solution

$$\begin{aligned}
& \log_{10}(5x+10) + 2\log_{10} 3 = 1 + \log_{10}(4x+12) \\
\Rightarrow\quad & \log_{10}(5x+10) + \log_{10} 3^2 - \log_{10}(4x+12) = 1 \\
\Rightarrow\quad & \log_{10}(5x+10) + \log_{10} 9 - \log_{10}(4x+12) = 1 \\
\Rightarrow\quad & \log_{10}\left[\frac{9(5x+10)}{4x+12}\right] = 1 \\
\Rightarrow\quad & \frac{9(5x+10)}{4x+12} = 10 \\
\Rightarrow\quad & 9(5x+10) = 10(4x+12) \\
\Rightarrow\quad & 45x+90 = 40x+120 \\
\Rightarrow\quad & 5x = 30 \\
\Rightarrow\quad & \underline{\underline{x=6}}.
\end{aligned}$$

(b) (3)

$$\frac{9^{2y}}{3^{7-y}} = \frac{3^{4y+3}}{27^{y-2}}.$$

Solution

$$\begin{aligned}
& \frac{9^{2y}}{3^{7-y}} = \frac{3^{4y+3}}{27^{y-2}} \\
\Rightarrow\quad & \frac{(3^2)^{2y}}{3^{7-y}} = \frac{3^{4y+3}}{(3^3)^{y-2}} \\
\Rightarrow\quad & \frac{3^{2(2y)}}{3^{7-y}} = \frac{3^{4y+3}}{3^{3(y-2)}} \\
\Rightarrow\quad & \frac{3^{4y}}{3^{7-y}} = \frac{3^{4y+3}}{3^{3y-6}} \\
\Rightarrow\quad & 3^{4y-(7-y)} = 3^{(4y+3)-(3y-6)} \\
\Rightarrow\quad & 3^{5y-7} = 3^{y+9}
\end{aligned}$$

take $\log_3$:

$$\begin{aligned}
\Rightarrow\quad & \log_3 3^{5y-7} = \log_3 3^{y+9} \\
\Rightarrow\quad & 5y-7 = y+9 \\
\Rightarrow\quad & 4y = 16 \\
\Rightarrow\quad & \underline{\underline{y=4}}.
\end{aligned}$$

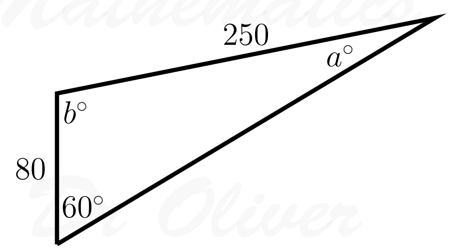
9. A plane, whose speed in still air is 250 kmh^{-1}, flies directly from A to B, where B is 500 km from A on a bearing of 060°. (7)

There is a constant wind of 80 kmh^{-1} blowing from the south.

Find, to the nearest minute, the time taken for the flight.

Solution

Well, the flight looks like this:



Sine rule:

$$\frac{\sin a^\circ}{80} = \frac{\sin 60^\circ}{250} \Rightarrow \sin a^\circ = \frac{80 \sin 60^\circ}{250}$$
$$\Rightarrow a = 16.088\,877 \text{ (FCD)}$$
$$\Rightarrow b = 103.911\,123 \text{ (FCD)}.$$

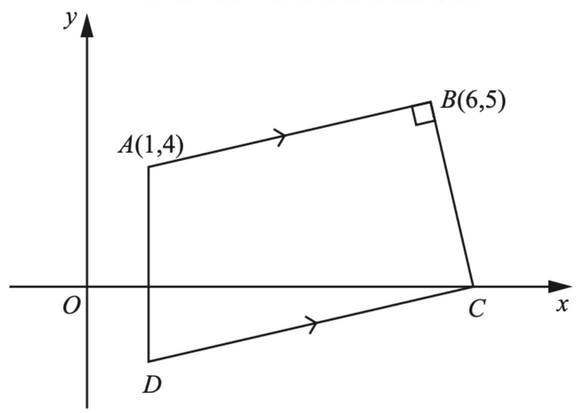
Cosine rule:

$$v^2 = 80^2 + 250^2 - 2 \times 80 \times 250 \times \cos 103.911\ldots^\circ$$
$$\Rightarrow \quad v^2 = 78\,516.659\,44 \text{ (FCD)}$$
$$\Rightarrow \quad v = 280.208\,243 \text{ (FCD)}.$$

Finally,

$$\text{time taken} = \frac{500}{280.208\ldots}$$
$$= 1.784\,387\,192 \text{ (FCD)}$$
$$= \underline{\underline{1 \text{ hour } 47 \text{ minutes (nearest minute)}}}.$$

10. **Solutions to this question by accurate drawing will not be accepted.**
The diagram shows a quadrilateral $ABCD$ in which A is the point $(1, 4)$ and B is the point $(6, 5)$.



- Angle ABC is a right angle and the point C lies on the x-axis.
- The line AD is parallel to the y-axis and the line CD is parallel to BA.

Find

(a) the equation of the line CD, (5)

Solution

Well,

$$\begin{aligned} m_{CD} &= m_{AB} \\ &= \frac{5-4}{6-1} \\ &= \tfrac{1}{5} \end{aligned}$$

and

$$m_{BC} = -5.$$

Now, the equation of BC is

$$y - 5 = -5(x - 6)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -5 = -5(x-6) \\ &\Rightarrow x - 6 = 1 \\ &\Rightarrow x = 7; \end{aligned}$$

so, $C(7, 0)$. Finally, the equation of CD is

$$y - 0 = \tfrac{1}{5}(x - 7) \Rightarrow \underline{\underline{y = \tfrac{1}{5}x - \tfrac{7}{5}}}.$$

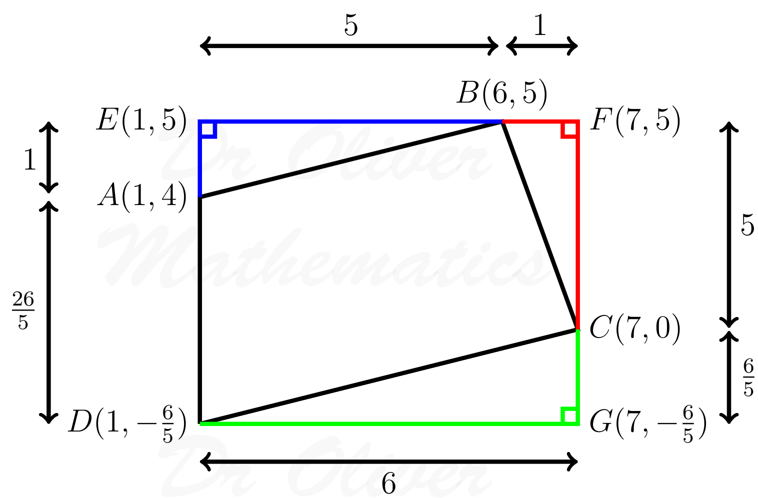
(b) the area of the quadrilateral $ABCD$. (4)

Solution
Well,

$$\begin{aligned} x = 1 &\Rightarrow y = \tfrac{1}{5} - \tfrac{7}{5} \\ &\Rightarrow y = -\tfrac{6}{5}, \end{aligned}$$

and $D(1, -\frac{6}{5})$.

Let $E(1, 5)$, $F(7, 5)$, and $G(7, -\frac{6}{5})$. A quick sketch:



Then

$$\begin{aligned} & \text{area of the quadrilateral } ABCD \\ = \; & \text{area of } EFGD - \text{area of } AED - \text{area of } BCF - \text{area of } CDG \\ = \; & (\tfrac{31}{5} \times 6) - (\tfrac{1}{2} \times 1 \times 5) - (\tfrac{1}{2} \times 1 \times 5) - (\tfrac{1}{2} \times 6 \times \tfrac{6}{5}) \\ = \; & 37\tfrac{1}{5} - \tfrac{5}{2} - \tfrac{5}{2} - 3\tfrac{3}{5} \\ = \; & \underline{\underline{28.6}}. \end{aligned}$$

11. Solve the equation

(a) $5 \sin x - 3 \cos x = 0$, for $0^\circ \leqslant x \leqslant 360^\circ$, (3)

Solution

Well,

$$
\begin{aligned}
5\sin x - 3\cos x = 0 &\Rightarrow 5\sin x = 3\cos x \\
&\Rightarrow \tan x = \tfrac{3}{5} \\
&\Rightarrow x = 30.963\,756\,3,\ 210.963\,756\,5 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = 31.0,\ 211 \text{ (3 sf)}}}.
\end{aligned}
$$

(b) $2\cos^2 y - \sin y - 1 = 0$, $0^\circ \leqslant y \leqslant 360^\circ$, (5)

Solution

Now,

$$
\begin{aligned}
2\cos^2 y - \sin y - 1 = 0 &\Rightarrow 2(1 - \sin^2 y) - \sin y - 1 = 0 \\
&\Rightarrow 2 - 2\sin^2 y - \sin y - 1 = 0 \\
&\Rightarrow -2\sin^2 y - \sin y + 1 = 0 \\
&\Rightarrow 2\sin^2 y + \sin y - 1 = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll}
\text{add to:} & +1 \\
\text{multiply to:} & (+2) \times (-1) = -2
\end{array}\right\} +2,\ -1
$$

e.g.,

$$
\begin{aligned}
&\Rightarrow 2\sin^2 y + 2\sin y - \sin y - 1 = 0 \\
&\Rightarrow 2\sin y(\sin y + 1) - 1(\sin y + 1) = 0 \\
&\Rightarrow (2\sin y - 1)(\sin y + 1) = 0 \\
&\Rightarrow \sin y = \tfrac{1}{2} \text{ or } \sin y = -1.
\end{aligned}
$$

$\underline{\sin y = \tfrac{1}{2}}$:

$$\sin y = \tfrac{1}{2} \Rightarrow \underline{\underline{y = 30, 150}}.$$

$\underline{\sin y = -1}$:

$$\sin y = -1 \Rightarrow \underline{\underline{y = 270}}.$$

(c) $3\sec z = 10$, for $0 \leqslant z \leqslant 6$ radians. (3)

Solution

$$\begin{aligned} 3\sec z = 10 &\Rightarrow \sec z = \tfrac{10}{3} \\ &\Rightarrow \cos z = \tfrac{3}{10} \\ &\Rightarrow z = 1.266\,103\,673,\ 5.017\,081\,634 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{z = 1.27,\ 5.02 \text{ (3 sf)}}}. \end{aligned}$$

EITHER

12. The functions f and g are defined, for $x > 1$, by

$$\begin{aligned} \mathrm{f}(x) &= (x+1)^2 - 4, \\ \mathrm{g}(x) &= \frac{3x+5}{x-1}. \end{aligned}$$

Find

(a) $\mathrm{f}\,\mathrm{g}(9)$, (2)

Solution

$$\begin{aligned} \mathrm{f}\,\mathrm{g}(9) &= \mathrm{f}(\mathrm{g}(9)) \\ &= \mathrm{f}(4) \\ &= \underline{\underline{21}}. \end{aligned}$$

(b) expressions for $\mathrm{f}^{-1}(x)$ and $\mathrm{g}^{-1}(x)$, (4)

Solution

$\underline{\mathrm{f}^{-1}(x)}$:

$$\begin{aligned} y = (x+1)^2 - 4 &\Rightarrow y + 4 = (x+1)^2 \\ &\Rightarrow \sqrt{y+4} = x + 1 \\ &\Rightarrow \sqrt{y+4} - 1 = x \end{aligned}$$

and so

$$\underline{\underline{\mathrm{f}^{-1}(x) = \sqrt{x+4} - 1}}.$$

$\underline{g^{-1}(x)}$:

$$y = \frac{3x+5}{x-1} \Rightarrow y(x-1) = 3x+5$$
$$\Rightarrow xy - y = 3x + 5$$
$$\Rightarrow xy - 3x = y + 5$$
$$\Rightarrow x(y-3) = y + 5$$
$$\Rightarrow x = \frac{y+5}{y-3}$$

and so

$$\underline{\underline{g^{-1}(x) = \frac{x+5}{x-3}}}.$$

(c) the value of x for which (4)

$$g(x) = g^{-1}(x).$$

Solution

Now,

$$g(x) = g^{-1}(x) \Rightarrow \frac{3x+5}{x-1} = \frac{x+5}{x-3}$$

cross-multiply:

$$\Rightarrow (3x+5)(x-3) = (x-1)(x+5)$$

$\times$	$3x$	$+5$
x	$3x^2$	$+5x$
-3	$-9x$	-15

$\times$	x	-1
x	x^2	$-x$
$+5$	$+5x$	-5

$$\begin{aligned}&\Rightarrow 3x^2 - 4x - 15 = x^2 + 4x - 5\\ &\Rightarrow 2x^2 - 8x - 10 = 0\\ &\Rightarrow 2(x^2 - 4x - 5) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -4\\ \text{multiply to:} & -5\end{array}\right\} -5,\ +1$$

$$\begin{aligned}&\Rightarrow 2(x-5)(x+1) = 0\\ &\Rightarrow x = 5 \text{ or } x = -1.\end{aligned}$$

But $x > 1$! So,

$$\underline{\underline{x = 5}}.$$

OR

13. A particle moves in a straight line so that, at time t s after passing a fixed point O, its velocity is v ms^{-1}, where

$$v = 6t + 4\cos 2t.$$

Find

(a) the velocity of the particle at the instant it passes O, (1)

Solution
Well,

$$t = 0 \Rightarrow \underline{\underline{v = 4 \text{ ms}^{-1}}}.$$

(b) the acceleration of the particle when $t = 5$, (4)

Solution
Now,

$$v = 6t + 4\cos 2t \Rightarrow a = 6 - 8\sin 2t$$

and

$$\begin{aligned} t = 5 \Rightarrow a &= 6 - 8\sin 10 \\ \Rightarrow a &= 10.352\,168\,89 \text{ (FCD)} \\ \Rightarrow \underline{\underline{a}} &\underline{\underline{= 10.4 \text{ ms}^{-2} \text{ (3 sf)}}}. \end{aligned}$$

(c) the greatest value of the acceleration, (1)

Solution
Well, the greatest value of the acceleration is

$$6 + 8 = \underline{\underline{14 \text{ ms}^{-2}}}.$$

(d) the distance travelled in the fifth second. (4)

Solution
Now,

$$\begin{aligned} \text{distance travelled in the fifth second} &= \int_4^5 (6t + 4\cos 2t)\,\mathrm{d}t \\ &= \left[3t^2 + 2\sin 2t\right]_{t=4}^5 \\ &= (75 + 2\sin 10) - (48 + 2\sin 8) \\ &= 23.933\,241\,28 \text{ (FCD)} \\ &= \underline{\underline{23.9 \text{ m (3 sf)}}}. \end{aligned}$$