Dr Oliver Mathematics
Further Mathematics
Second Order Differential Equations
Past Examination Questions

This booklet consists of 37 questions across a variety of examination topics.
The total number of marks available is 435.

1. Find the general solution of the differential equation (6)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x}.$$

2. Find the general solution of the differential equation (7)

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 4x.$$

3. (a) Find the general solution of the differential equation (7)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 17x + 36.$$

   (b) Show that, when x is large and positive, the solution approximates to a linear function and state the equation of the linear function. (2)

4. Find the general solution of the differential equation (9)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 65\sin 2x.$$

5. The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}.$$

   (a) Find the complementary function. (3)

   (b) Explain briefly why there is no particular integral if either of the forms $y = ke^{3x}$ or $y = kxe^{3x}$. (1)

   (c) Given that there is a particular integral of the form $y = kx^2e^{3x}$, find the value of k. (5)

6. Solve the differential equation (10)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\mathrm{e}^{-x}$$

given that $y \to 0$ as $x \to \infty$ and that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3$ when $x = 0$.

7. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} = 12\mathrm{e}^{2x}.$$

(a) Find the general solution of the differential equation. (6)

(b) It is given that the curve which represents a particular solution of the differential equation has gradient 6 when $x = 0$ and approximates to $y = \mathrm{e}^{2x}$ when x is large and positive. Find the equation of the curve. (4)

8. A differential equation is given by

$$\sin^2 x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sin x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x\cos x,\ 0 < x < \pi.$$

(a) Show that the substitution $y = u\sin x$, where u is a function of x, transforms this differential equation into (5)

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + u = \sin 2x.$$

(b) Hence find the general solution to the differential equation (6)

$$\sin^2 x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sin x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x\cos x$$

giving your answer in the form $y = \mathrm{f}(x)$.

9. The differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin kx$$

is to be solved, where k is a constant.

(a) In the case $k = 2$, by using a particular integral of the form $ax\cos 2x + bx\sin 2x$, find the general solution. (7)

(b) Describe briefly the behaviour of your solution for y when $x \to \infty$. (2)

(c) In the case $k \neq 2$, explain briefly whether y would exhibit the same behaviour as in part (b) when $x \to \infty$. (2)

10. The variables x and y satisfy the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 5\mathrm{e}^{-2x}.$$

(a) Find the complementary function of the differential equation. (2)

(b) Given that there is a particular integral of the form $y = px\mathrm{e}^{-2x}$, find the constant p. (4)

(c) Find the solution of the differential equation for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$. (5)

11. Find the solution of the differential equation (11)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \mathrm{e}^{-x}$$

for which $y = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

12. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 8\cos 4x.$$

(a) Find the complementary function of the differential equation. (2)

(b) Given that there is a particular integral of the form $y = px\sin 4x$, where p is a constant, find the general solution of the differential equation. (6)

(c) Find the solution of the equation for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (4)

13. (a) Find the general solution of the differential equation (7)

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -2x + 13.$$

(b) Find the particular solution for which $y = -\frac{7}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (5)

(c) Write down the function to which y approximates when x is large and positive. (1)

14. (a) Find the complementary function of the differential equation (2)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \operatorname{cosec} x.$$

(b) It is given that

$$y = p(\ln \sin x)\sin x + qx\cos x,$$

where p and q are constants, is a particular integral of the differential equation.

(i) Show that (6)

$$p - 2(p+q)\sin^2 x \equiv 1.$$

(ii) Deduce the values of p and q. (2)

(c) Write down the general solution of the differential equation. State the set of values of x, in the interval $0 \leqslant x \leqslant 2\pi$, for which the solution is valid, justifying your answer. (3)

15. (a) Find the general solution of the differential equation (6)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t},$$

giving your answer in the form $y = \mathrm{f}(t)$.

(b) Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$, and y is a function of x, show that (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t}.$$

(c) Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation (2)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t}.$$

(d) Hence write down the general solution of the differential equation (1)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}.$$

16. (a) Show that the substitution $x = \mathrm{e}^t$ transforms the differential equation (7)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 30 + 20\sin(\ln x)$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 3 + 20\sin t.$$

(b) Find the general solution of (11)

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3 + 20\sin t.$$

(c) Write down the general solution of (1)

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3 + 20\sin(\ln x).$$

17. (a) Show that the transformation $y = vx$ transforms the equation (5)

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + 9x^2)y = x^5 \qquad (\dagger)$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad (\ddagger)$$

(b) Solve the differential equation ($\ddagger$) to find v as a function of x. (6)

(c) Hence state the general solution of the differential equation ($\dagger$). (1)

18. (a) Find the general solution of the differential equation (6)

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9.$$

(b) Find the particular solution of this differential equation for which $x = 3$ and $\frac{dx}{dt} = -1$ when $t = 0$. (4)

The particular solution in part (b) is used to model the motion of a particle P on the x-axis. At time t seconds ($t \geqslant 0$), P is x metres from the origin O.

(c) Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum. (4)

19. Given that $3x\sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = k\cos 2x,$$

where k is a constant,

(a) calculate the value of k, (4)

(b) find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$. (4)

20. A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at a time t hours satisfies the differential equation

$$2x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = x^2 - 3x^4,\ x > 0.$$

(a) Show that the substitution $y = \dfrac{1}{x^2}$ transforms this differential equation into (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 3. \quad (\dagger)$$

(b) Find the general solution of the differential equation (†). (4)

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$,

(c) find an expression for x in terms of t, (4)

(d) write down the maximum values of x as t varies. (1)

21. For the differential equation (12)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x(x+3),$$

find the solution for which $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$, and $y = 1$.

22. (a) Find the general solution of the differential equation (8)

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2.$$

(b) Find the particular solution for which, at $x = 0$, $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$. (6)

23. (a) Find, in terms of k, the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = kt + 5,$$

where k is a constant and $t > 0$.

For large values of t, this general solution may be approximated by a linear function.

(b) Given that $k = 6$, find the equation of this linear function. (2)

24.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 4\mathrm{e}^x.$$

(a) Show that $\lambda x\mathrm{e}^x$ is a particular integral of the differential equation, where λ is a constant to be found. (4)

(b) Find general solution of the differential equation. (4)

(c) Find the particular solution for which $y = -\frac{2}{3}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}$ at $x = 0$. (5)

25. Find the general solution of the differential equation (8)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = \mathrm{e}^{-4t}.$$

26. Find the general solution of the differential equation (10)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 5\cos t.$$

27.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 2\mathrm{e}^{-t}.$$

Given that $x = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2$ at $t = 0$,

(a) find x in terms of t. (8)

The particular solution in part (a) is used to model the motion of a particle P on the x-axis. At time t seconds, where $t \geqslant 0$, P is x metres from the origin O.

(b) Show that the maximum distance between O and P is $\frac{2\sqrt{3}}{9}$ m and justify that the distance is a maximum. (7)

28. (a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 25y = 3\cos 5x.$$

(b) Using your answer to part (a), the general solution of the differential equation (3)

Given that at $x = 0$, $y = 0$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = \mathrm{f}(x)$. (5)

(d) Sketch the curve with equation $y = \mathrm{f}(x)$ for $0 \leqslant x \leqslant \pi$. (2)

29. The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = \cos 3t,\ t \geqslant 0,$$

describes the motion of a particle along the x-axis.

(a) Find the general solution to this differential equation. (8)

(b) Find the particular solution of this differential equation for which, at $t = 0$, $x = \frac{1}{2}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$. (5)

On the graph of the particular solution defined in part (b), the first turning point for $t > 30$ is the point A.

(c) Find the approximate values for the coordinates of A. (2)

30. Find the general solution to the differential equation (9)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 2\cos t - \sin t.$$

31. (a) Find the value of λ for which $\lambda t^2 \mathrm{e}^{3t}$ is a particular integral of the differential equation (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 6\mathrm{e}^{3t}, t \geqslant 0.$$

(b) Hence find the general solution of the differential equation. (3)

Given that when $t = 0$, $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 4$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = \mathrm{f}(t)$. (5)

32. (a) Show that the transformation $y = vx$ transforms the equation (6)

$$4x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8x\frac{\mathrm{d}y}{\mathrm{d}x} + (8 + 4x^2)y = x^4 \qquad (\dagger)$$

into the equation

$$4\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 4v = x. \qquad (\ddagger)$$

(b) Solve the differential equation (‡) to find v as a function of x. (6)

(c) Hence state the general solution of the differential equation (†). (1)

33. (a) Show that the substitution $x = \mathrm{e}^z$ transforms the differential equation (7)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\ln x, x > 0, \qquad (\dagger)$$

into the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 3z. \qquad (\ddagger)$$

(b) Find the general solution of the differential equation (‡). (6)

(c) Hence obtain the general solution of the differential equation (†) giving your answer in the form $y = \mathrm{f}(x)$. (1)

34. (a) Find the general solution of the differential equation (6)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 27\mathrm{e}^{-x}.$$

(b) Find the particular solution that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (6)

35. (a) Show that the transformation $x = \mathrm{e}^u$ transforms the differential equation (6)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 2\ln x,\ x > 0, \qquad \text{(I)}$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 8\frac{\mathrm{d}y}{\mathrm{d}u} + 16y = 2u \qquad \text{(II)}.$$

(b) Find the general solution of the differential equation (II), expressing y as a function of u. (7)

(c) Hence obtain the general solution of the differential equation (I). (1)

36. (a) Show that the transformation $x = \mathrm{e}^u$ transforms the differential equation (6)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = -x^{-2},\ x > 0, \qquad \text{(I)}$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 3\frac{\mathrm{d}y}{\mathrm{d}u} + 2y = -\mathrm{e}^{-2u} \qquad \text{(II)}.$$

(b) Find the general solution of the differential equation (II). (7)

(c) Hence obtain the general solution of the differential equation (I) giving your answer in the form $y = \mathrm{f}(x)$. (1)

37. (a) Find the general solution of the differential equation (8)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 26\sin 3x.$$

(b) Find the particular solution of this differential equation for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (5)