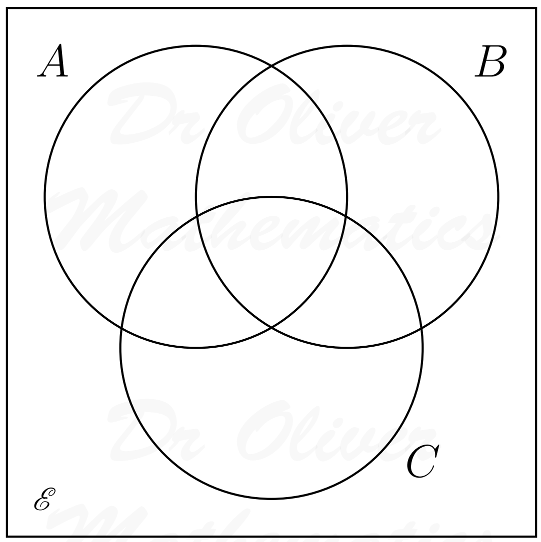
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2008 November Paper 1: Calculator
2 hours

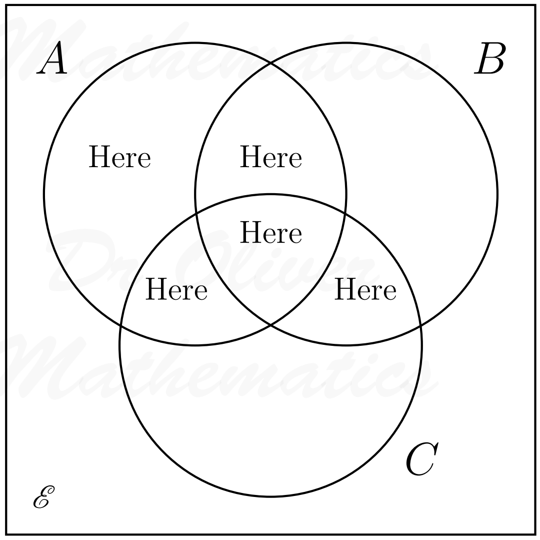
The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Here is a Venn diagram.



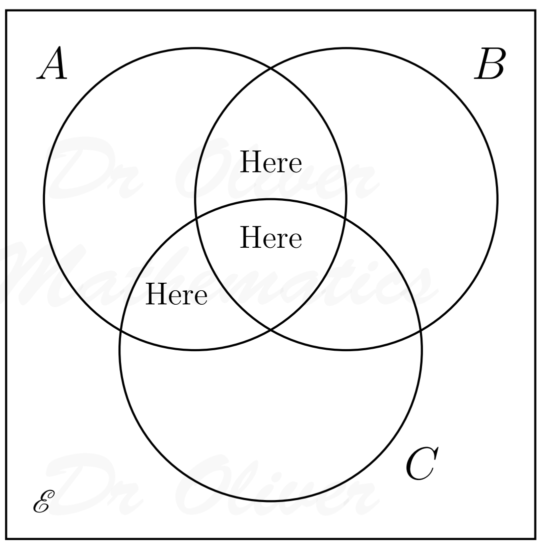
(a) Copy the Venn diagram above and shade the region that represents $A \cup (B \cap C)$. (1)

Solution



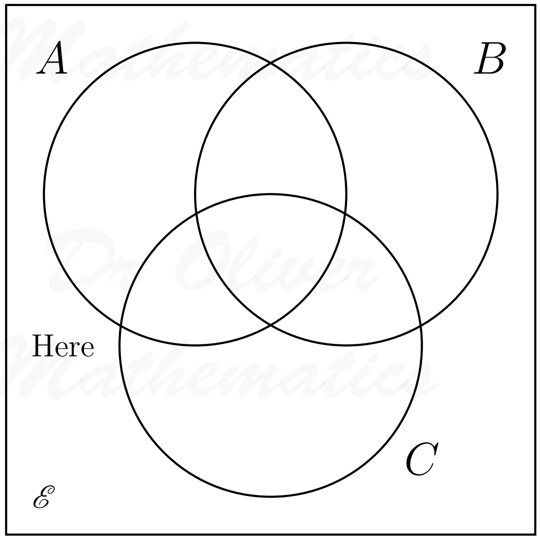
(b) Copy the Venn diagram above and shade the region that represents $A \cap (B \cup C)$. (1)

Solution



(c) Copy the Venn diagram above and shade the region that represents $(A \cup B \cup C)'$. (1)

Solution



2. Find the set of values of x for which (4)

$$(2x+1)^2 > 8x+9.$$

Solution

Well,

$\times$	$2x$	$+1$
$2x$	$4x^2$	$+2x$
$+1$	$+2x$	$+1$

so

$$\begin{aligned}(2x+1)^2 > 8x+9 &\Rightarrow 4x^2+4x+1 > 8x+9\\ &\Rightarrow 4x^2-4x-8 > 0\\ &\Rightarrow 4(x^2-x-1) > 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -1\\ \text{multiply to:} & -1\end{array}\right\} +1,\ -2$$

$$\Rightarrow 4(x+1)(x-2) > 0.$$

We need a 'table of signs':

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$x + 1$	$-$	0	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	0	$+$
$(x+1)(x-2)$	$+$	0	$-$	0	$+$

Hence,

$$(2x+1)^2 > 8x + 9 \Rightarrow \underline{\underline{x < -1 \text{ or } x > 2}}.$$

3. Prove that (4)

$$\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} \equiv 2 \operatorname{cosec} A.$$

Solution

$$\begin{aligned}
\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} &\equiv \frac{\sin^2 A + (1 + \cos A)^2}{\sin A(1 + \cos A)} \\
&\equiv \frac{\sin^2 A + (1 + 2\cos A + \cos^2 A)}{\sin A(1 + \cos A)} \\
&\equiv \frac{1 + 2\cos A + (\sin^2 A + \cos^2 A)}{\sin A(1 + \cos A)} \\
&\equiv \frac{1 + 2\cos A + 1}{\sin A(1 + \cos A)} \\
&\equiv \frac{2(1 + \cos A)}{\sin A(1 + \cos A)} \\
&\equiv \frac{2}{\sin A} \\
&\equiv \underline{\underline{2 \operatorname{cosec} A}},
\end{aligned}$$

as required.

4. A function f is such that (5)

$$f(x) = ax^3 + bx^2 + 3x + 4.$$

- When $f(x)$ is divided by $(x-1)$, the remainder is 3.
- When $f(x)$ is divided by $(2x+1)$, the remainder is 6.

Find the value of a and of b.

Solution

We use synthetic division twice:

$$\begin{array}{c|cccc} 1 & a & b & 3 & 4 \\ & \downarrow & a & a+b & a+b+3 \\ \hline & a & a+b & a+b+3 & a+b+7 \end{array}$$

so

$$a+b+7=3 \Rightarrow \boxed{a+b=-4 \quad (1).}$$

$$\begin{array}{c|cccc} -\frac{1}{2} & a & b & 3 & 4 \\ & \downarrow & -\frac{1}{2}a & \frac{1}{4}a-\frac{1}{2}b & -\frac{1}{8}a+\frac{1}{4}b-\frac{3}{2} \\ \hline & a & -\frac{1}{2}a+b & \frac{1}{4}a-\frac{1}{2}b+3 & -\frac{1}{8}a+\frac{1}{4}b+\frac{5}{2} \end{array}$$

so

$$-\tfrac{1}{8}a+\tfrac{1}{4}b+\tfrac{5}{2}=6 \Rightarrow -\tfrac{1}{8}a+\tfrac{1}{4}b=\tfrac{7}{2}$$
$$\Rightarrow \boxed{-a+2b=28 \quad (2).}$$

Do (1) + (2):

$$3b=24 \Rightarrow \underline{\underline{b=8}}$$
$$\Rightarrow a+8=-4$$
$$\Rightarrow \underline{\underline{a=-12}}.$$

5. Given that $\mathbf{a}=5\mathbf{i}-12\mathbf{j}$ and that $\mathbf{b}=p\mathbf{i}+\mathbf{j}$, find
   (a) the unit vector in the direction of $\mathbf{a}$, (2)

Solution

$$\begin{aligned}\sqrt{5^2 + (-12)^2} &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13\end{aligned}$$

so the unit vector in the direction of $\mathbf{a}$ is

$$\underline{\underline{\tfrac{1}{13}(5\mathbf{i} - 12\mathbf{j})}}.$$

(b) the values of the constants p and q such that (3)

$$q\mathbf{a} + \mathbf{b} = 19\mathbf{i} - 23\mathbf{j}.$$

Solution

Well,

$$\begin{aligned}& q\mathbf{a} + \mathbf{b} = 19\mathbf{i} - 23\mathbf{j} \\ \Rightarrow\quad & q(5\mathbf{i} - 12\mathbf{j}) + (p\mathbf{i} + \mathbf{j}) = 19\mathbf{i} - 23\mathbf{j} \\ \Rightarrow\quad & (5q + p)\mathbf{i} + (-12q + 1)\mathbf{j} = 19\mathbf{i} - 23\mathbf{j}.\end{aligned}$$

Take the $\mathbf{j}$th component:

$$\begin{aligned}-12q + 1 = -23 &\Rightarrow -12q = -24 \\ &\Rightarrow \underline{\underline{q = 2}}.\end{aligned}$$

Take the $\mathbf{i}$th component:

$$\begin{aligned}5q + p = 19 &\Rightarrow 5(2) + p = 19 \\ &\Rightarrow 10 + p = 19 \\ &\Rightarrow \underline{\underline{p = 9}}.\end{aligned}$$

6. (a) Solve the equation (3)

$$2t = 9 + \frac{5}{t}.$$

Solution

Multiply each side by t:

$$2t = 9 + \frac{5}{t} \Rightarrow 2t^2 = 9t + 5$$
$$\Rightarrow 2t^2 - 9t - 5 = 0$$

e.g.,

$$\left.\begin{array}{ll} \text{add to:} & -9 \\ \text{multiply to:} & (+2) \times (-5) = -10 \end{array}\right\} -10,\ +1$$

$$\Rightarrow 2t^2 - 10t + t - 5 = 0$$
$$\Rightarrow 2t(t-5) + 1(t-5) = 0$$
$$\Rightarrow (2t+1)(t-5) = 0$$
$$\Rightarrow 2t + 1 = 0 \text{ or } t - 5 = 0$$
$$\Rightarrow \underline{\underline{t = -\tfrac{1}{2} \text{ or } t = 5}}.$$

(b) Hence, or otherwise, solve the equation (3)

$$2x^{\frac{1}{2}} = 9 + 5x^{-\frac{1}{2}}.$$

Solution

$$2x^{\frac{1}{2}} = 9 + 5x^{-\frac{1}{2}} \Rightarrow x^{\frac{1}{2}} = (-\tfrac{1}{2})^2 \text{ or } x^{\frac{1}{2}} = 5^2$$
$$\Rightarrow \underline{\underline{x = \tfrac{1}{4} \text{ or } x = 25}}.$$

7. (a) Express (3)

$$4x^2 - 12x + 3$$

in the form

$$(ax+b)^2 + c,$$

where a, b, and c are constants and $a > 0$.

Solution

$$\begin{aligned}4x^2 - 12x + 3 &\Rightarrow (4x^2 - 12x) + 3\\ &\Rightarrow [(4x^2 - 12x + 9) - 9] + 3\\ &\Rightarrow (2x-3)^2 - 9 + 3\\ &\Rightarrow \underline{\underline{(2x-3)^2 - 6}};\end{aligned}$$

hence, $\underline{\underline{a = 2}}$, $\underline{\underline{b = -3}}$, and $\underline{\underline{c = -6}}$.

(b) Hence, or otherwise, find the coordinates of the stationary point of the curve. (2)

Solution

$\underline{\underline{(\frac{3}{2}, -6)}}$.

(c) Given that (1)

$$f(x) = 4x^2 - 12x + 3,$$

write down the range of f.

Solution

$\underline{\underline{f(x) \geqslant -6}}$.

8. A curve is such that (6)

$$\frac{d^2y}{dx^2} = 4e^{-2x}.$$

Given that

$$\frac{dy}{dx} = 3$$

when $x = 0$ and that the curve passes through the point $(2, e^{-4})$, find the equation of the curve.

Solution

Well,

$$\frac{d^2y}{dx^2} = 4e^{-2x} \Rightarrow \frac{dy}{dx} = -2e^{-2x} + c,$$

for some constant c. Now,

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 3 &\Rightarrow -2\mathrm{e}^0 + c = 3 \\ &\Rightarrow -2 + c = 3 \\ &\Rightarrow c = 5, \end{aligned}$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\mathrm{e}^{-2x} + 5.$$

Next,

$$y = \mathrm{e}^{-2x} + 5x + d,$$

for some constant d:

$$\begin{aligned} x = 2, y = \mathrm{e}^{-4} &\Rightarrow \mathrm{e}^{-2(2)} + 5(2) + d = \mathrm{e}^{-4} \\ &\Rightarrow \mathrm{e}^{-4} + 10 + d = \mathrm{e}^{-4} \\ &\Rightarrow d = -10; \end{aligned}$$

hence,

$$\underline{\underline{y = \mathrm{e}^{-2x} + 5x - 10}}.$$

9. (a) Find, in ascending powers of x, the first 3 terms in the expansion of (3)

$$(2 - 3x)^5.$$

Solution

$$\begin{aligned} (2 - 3x)^5 &= (2)^5 + \binom{5}{1}(2)^4(-3x)^1 + \binom{5}{2}(2)^3(-3x)^2 + \ldots \\ &= \underline{\underline{32 - 240x + 720x^2 + \ldots}}. \end{aligned}$$

The first 3 terms in the expansion of

$$(a + bx)(2 - 3x)^5$$

in ascending powers of x are are

$$64 - 192x + cx^2.$$

(b) Find the value of a, of b, and of c. (5)

Solution

$\times$	32	$-240x$	$+720x^2$
a	$32a$	$-240ax$	$+720ax^2$
$+bx$	$+32bx$	$-240bx^2$	$\ldots$

so

$$(a+bx)(2-3x)^5 = 32a + (-240a + 32b)x + (720a - 240b)x^2 + \ldots.$$

Solving:

$$\begin{aligned}
a:&\quad 32a = 64 \\
&\quad \Rightarrow \underline{\underline{a = 2}}, \\
b:&\quad -240a + 32b = -192 \\
&\quad \Rightarrow -480 + 32b = -192 \\
&\quad \Rightarrow 32b = 288 \\
&\quad \Rightarrow \underline{\underline{b = 9}}, \\
c:&\quad c = 720(2) - 240(9) \\
&\quad \Rightarrow c = 1\,440 - 2\,160 \\
&\quad \Rightarrow \underline{\underline{c = -720}}.
\end{aligned}$$

10. Functions f and g are defined, for $x \in \mathbb{R}$, by

$$\begin{aligned}
\mathrm{f}(x) &= 3 - x, \\
\mathrm{g}(x) &= \frac{x}{x+2}, \text{ where } x \neq -2.
\end{aligned}$$

(a) (i) Find $\mathrm{f\,g}(x)$. (2)

Solution

$$\begin{aligned} \mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}\left(\frac{x}{x+2}\right) \\ &= \underline{\underline{3 - \frac{x}{x+2}}}. \end{aligned}$$

(ii) Hence find the value of x for which (2)

$$\mathrm{f\,g}(x) = 10.$$

Solution

$$\begin{aligned} \mathrm{f\,g}(x) = 10 &\Rightarrow 3 - \frac{x}{x+2} = 10 \\ &\Rightarrow -7 = \frac{x}{x+2} \\ &\Rightarrow -7(x+2) = x \\ &\Rightarrow -7x - 14 = x \\ &\Rightarrow -14 = 8x \\ &\Rightarrow \underline{\underline{x = -1\tfrac{3}{4}}}. \end{aligned}$$

(b) A function h is defined, for $x \in \mathbb{R}$, by

$$\mathrm{h}(x) = 4 + \ln x, \text{ where } x > 1.$$

(i) Find the range of h. (1)

Solution
$\underline{\underline{\mathrm{h}(x) > 4}}$.

(ii) Find the value of $\mathrm{h}^{-1}(9)$. (2)

Solution
Well,

$$\begin{aligned} y = 4 + \ln x &\Rightarrow y - 4 = \ln x \\ &\Rightarrow \mathrm{e}^{y-4} = x \end{aligned}$$

and so

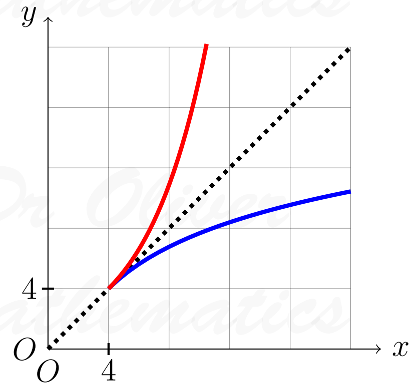
$$\mathrm{h}^{-1}(x) = \mathrm{e}^{x-4}.$$

Finally,

$$\mathrm{h}^{-1}(9) = \mathrm{e}^{9-4} = \underline{\underline{\mathrm{e}^5}}.$$

(iii) On the same axes, sketch the graphs of $y = \mathrm{h}(x)$ and $y = \mathrm{h}^{-1}(x)$. (3)

Solution



11. Solve the equation

(a) $\tan 2x - 3\cot 2x = 0$, for $0° < x < 180°$, (4)

Solution

Now,

$$0° < x < 180° \Rightarrow 0° < 2x < 360°$$

and so

$$\begin{aligned} \tan 2x - 3\cot 2x = 0 &\Rightarrow \tan 2x = 3\cot 2x \\ &\Rightarrow \tan 2x = \frac{3}{\tan 2x} \\ &\Rightarrow \tan^2 2x = 3 \\ &\Rightarrow \tan 2x = \pm\sqrt{3}. \end{aligned}$$

$\underline{\tan 2x = \sqrt{3}}$:

$$\begin{aligned} \tan 2x = \sqrt{3} &\Rightarrow 2x = 60,\ 240 \\ &\Rightarrow \underline{\underline{x = 30,\ 120}}. \end{aligned}$$

$\underline{\tan 2x = -\sqrt{3}}$:

$$\tan 2x = \sqrt{3} \Rightarrow 2x = 120,\ 300$$
$$\Rightarrow \underline{\underline{x = 60,\ 150}}.$$

(b) $\operatorname{cosec} y = 1 - 2\cot^2 y$, for $0^\circ < y < 360^\circ$, (5)

Solution

$$\begin{aligned}\operatorname{cosec} y = 1 - 2\cot^2 y &\Rightarrow \operatorname{cosec} y = 1 - 2(\operatorname{cosec}^2 y - 1)\\ &\Rightarrow \operatorname{cosec} y = 1 - 2\operatorname{cosec}^2 y + 2\\ &\Rightarrow 2\operatorname{cosec}^2 y + \operatorname{cosec} y - 3 = 0\end{aligned}$$

e.g.,

$$\left.\begin{array}{ll}\text{add to:} & +1\\ \text{multiply to:} & (+2)\times(-3) = -6\end{array}\right\} + 3,\ -2$$

$$\begin{aligned}&\Rightarrow 2\operatorname{cosec}^2 y + 3\operatorname{cosec} y - 2\operatorname{cosec} - 3 = 0\\ &\Rightarrow \operatorname{cosec} y(2\operatorname{cosec} y + 3) - 1(2\operatorname{cosec} + 3) = 0\\ &\Rightarrow (\operatorname{cosec} y - 1)(2\operatorname{cosec} y + 3) = 0\\ &\Rightarrow \operatorname{cosec} y = 1 \text{ or } \operatorname{cosec} y = -\tfrac{3}{2}\\ &\Rightarrow \sin y = 1 \text{ or } \sin y = -\tfrac{2}{3}.\end{aligned}$$

$\underline{\sin y = 1}$

$$\sin y = 1 \Rightarrow \underline{\underline{y = 90}}.$$

$\underline{\sin y = -\tfrac{2}{3}}$

$$\sin y = -\tfrac{2}{3} \Rightarrow y = 221.810\,314\,9,\ 318.189\,685\,1 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{y = 221,\ 318 \text{ (3 sf)}}}.$$

(c) $\sec(z + \frac{1}{2}\pi) = -2$, for $0 < z < \pi$ radians. (3)

Solution

$$0 < z + \tfrac{1}{2}\pi < \pi$$

$$\begin{aligned}\sec(z+\tfrac{1}{2}\pi) = -2 &\Rightarrow \cos(z+\tfrac{1}{2}\pi) = -\tfrac{1}{2}\\ &\Rightarrow z+\tfrac{1}{2}\pi = \tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi\\ &\Rightarrow \underline{\underline{z = \tfrac{1}{6}\pi,\ \tfrac{5}{6}\pi}}.\end{aligned}$$

EITHER

12. A curve has equation

$$y = \frac{x^2}{x+1}.$$

(a) Find the coordinates of the stationary points of the curve. (5)

Solution

Quotient rule:

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$v = x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+1)(2x)-(1)(x^2)}{(x+1)^2}\\ &= \frac{2x^2+2x-x^2}{(x+1)^2}\\ &= \frac{x^2+2x}{(x+1)^2}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{x^2+2x}{(x+1)^2} = 0\\ &\Rightarrow x(x+2) = 0\\ &\Rightarrow x = 0 \text{ or } x = -2\\ &\Rightarrow y = 0 \text{ or } y = -4;\end{aligned}$$

hence, the coordinates of the stationary points of the curve are

$$\underline{\underline{(0,0) \text{ and } (-2,-4)}}.$$

The normal to the curve at the point where $x = 1$ meets the x-axis at M.

The tangent to the curve at the point where $x = -2$ meets the y-axis at N.

(b) Find the area of the triangle MNO, where O is the origin. (6)

Solution

$\underline{x = 1}$:

$$\begin{aligned} x = 1 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{4} \\ &\Rightarrow m_{\text{normal}} = -\tfrac{4}{3} \end{aligned}$$

and

$$x = 1 \Rightarrow y = \tfrac{1}{2}.$$

Now, the normal to the curve at the point where $x = 1$ is

$$y - \tfrac{1}{2} = -\tfrac{4}{3}(x - 1)$$

and it meets the x-axis at

$$\begin{aligned} -\tfrac{1}{2} = -\tfrac{4}{3}(x-1) &\Rightarrow x - 1 = \tfrac{3}{8} \\ &\Rightarrow x = 1\tfrac{3}{8}; \end{aligned}$$

so, $M(1\frac{3}{8}, 0)$.

$\underline{x = -2}$:

$$x = -2 \Rightarrow y = -4, \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and the tangent to the curve at the point where $x = -2$ is

$$y = -4.$$

So, $N(0, -4)$.

Finally,

$$\begin{aligned} \text{area of the triangle } MNO &= \tfrac{1}{2} \times \tfrac{11}{8} \times 4 \\ &= \underline{\underline{2\tfrac{3}{4}}}. \end{aligned}$$

OR

13. A curve has equation

$$y = \mathrm{e}^{x-2} - 2x + 6.$$

(a) Find the coordinates of the stationary point of the curve and determine the nature of the stationary point. (6)

Solution

$$y = \mathrm{e}^{x-2} - 2x + 6 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x-2} - 2$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \mathrm{e}^{x-2} - 2 = 0 \\
&\Rightarrow \mathrm{e}^{x-2} = 2 \\
&\Rightarrow x - 2 = \ln 2 \\
&\Rightarrow x = 2 + \ln 2 \\
&\Rightarrow y = \mathrm{e}^{2+\ln 2-2} - 2(2 + \ln 2) + 6 \\
&\Rightarrow y = \mathrm{e}^{\ln 2} - 4 - 2\ln 2 + 6 \\
&\Rightarrow y = 2 + 2 - 2\ln 2 \\
&\Rightarrow y = 4 - 2\ln 2;
\end{aligned}$$

so, the coordinates of the stationary point is

$$\underline{\underline{(2 + \ln 2, 4 - 2\ln 2)}}.$$

Next,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^{x-2}$$

and

$$\begin{aligned}
x = 2 + \ln 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \mathrm{e}^{2+\ln 2-2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \mathrm{e}^{\ln 2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2 \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &> 0;
\end{aligned}$$

hence, the nature of the stationary point is a $\underline{\underline{\text{minimum}}}$.

The area of the region enclosed by the curve, the positive x-axis, the positive y-axis, and the line $x = 3$ is

$$k + \mathrm{e} - \mathrm{e}^{-2}.$$

(b) Find the value of k. (5)

Solution

$$\begin{aligned}\int_0^3 (\mathrm{e}^{x-2} - 2x + 6)\,\mathrm{d}x &\Rightarrow \left[\mathrm{e}^{x-2} - x^2 + 6x\right]_{x=0}^{3} \\ &\Rightarrow (\mathrm{e} - 9 + 18) - \left(\mathrm{e}^{-2} - 0 + 0\right) \\ &\Rightarrow 9 + \mathrm{e} - \mathrm{e}^{-2};\end{aligned}$$

so, $\underline{\underline{k = 9}}$.