Dr Oliver Mathematics
Direct Proof

In this note, we explore the idea of direct proof.

Example 1
The product of any two odd integers is odd.

Solution
Suppose m and n are any particular but arbitrarily chosen odd integers. By the definition of odd numbers, we have

$$n = 2r + 1$$

for some integer r and

$$m = 2s + 1$$

for some integer s. Then, by substitution, we have

$$\begin{aligned} mn &= (2r + 1)(2s + 1) \\ &= 4rs + 2r + 2s + 1 \\ &= 2(2rs + r + s) + 1. \end{aligned}$$

Now, $(2rs + r + s)$ is an integer (because products and sums of integers are integers and 2, r, and s are all integers) and therefore, by definition of odd number, mn is odd. $\blacksquare$

Example 2
For all integers n,

$$4(n^2 + n + 1) - 3n^2$$

is a perfect square.

Solution
Let n is any particular but arbitrarily chosen integer. Then

$$\begin{aligned} 4(n^2 + n + 1) - 3n^2 &= (4n^2 + 4n + 4) - 3n^2 \\ &= n^2 + 4n + 4 \\ &= (n + 2)^2. \end{aligned}$$

Now, $4(n^2 + n + 1) - 3n^2$ is a perfect square because $(n + 2)$ is an integer (being a sum of n and 2). $\blacksquare$

Example 3
For all integers a, b and c, if a divides b and b divides c, then a divides c.

Solution
Since a divides b,

$$b = ar \text{ for some integer } r$$

and, since b divides c,

$$c = bs \text{ for some integer } s.$$

Now,

$$\begin{aligned} c &= bs \\ &= (ar)s \\ &= a(rs) \end{aligned}$$

and a divides c. ■

Here are some examples for you to try.

1. For all integer a, b and c, if $a|b$ and $a|c$, then $a|(b+c)$.

> **Solution**
>
> Since a divides b,
>
> $$b = ar \text{ for some integer } r$$
>
> and, since a divides c,
>
> $$c = as \text{ for some integer } s.$$
>
> Now,
>
> $$\begin{aligned} b + c &= ar + as \\ &= a(r+s), \end{aligned}$$
>
> and $a|(b+c)$.

2. Suppose a, b, c, and $d \in \mathbb{Z}$. If $a|b$ and $c|d$, then $ac|bd$.

> **Solution**
>
> Since a divides b,
>
> $$b = ar \text{ for some integer } r$$
>
> and, since c divides d,
>
> $$d = cs \text{ for some integer } s.$$
>
> Now,
>
> $$\begin{aligned} bd &= (ar)(cs) \\ &= ac(rs), \end{aligned}$$
>
> and $ac|bd$.

3. If two integers have opposite parity, then their sum is odd.

Solution

Suppose a and b are two integers with opposite parity. Without loss of generality, suppose a is even and b is odd. Thus $c = 2a$ and $d = 2b + 1$ for some integers c and d. Therefore,

$$\begin{aligned} c + d &= 2a + (2b + 1) \\ &= 2(a + b) + 1, \end{aligned}$$

which is odd.

4. Let x and y be positive numbers. If $x \leqslant y$, then $\sqrt{x} \leqslant \sqrt{y}$.

Solution

Now,

$$\begin{aligned} x \leqslant y &\Rightarrow x - y \leqslant 0 \\ &\Rightarrow (\sqrt{x})^2 - (\sqrt{y})^2 \leqslant 0 \\ &\Rightarrow (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) \leqslant 0 \\ &\Rightarrow \sqrt{x} - \sqrt{y} \leqslant 0 \\ &\Rightarrow \sqrt{x} \leqslant \sqrt{y}. \end{aligned}$$

5. If a is an odd integer, then $a^2 + 3a + 5$ is odd.

Solution

If a is an odd integer, there exists an integer r such that $a = 2r + 1$. Now,

$$\begin{aligned} a^2 + 3a + 5 &= (2r + 1)^2 + 3(2r + 1) + 5 \\ &= (4r^2 + 4r + 1) + (6r + 3) + 5 \\ &= 4r^2 + 10r + 9 \\ &= 2(2r^2 + 5r + 4) + 1, \end{aligned}$$

and $a^2 + 3a + 5$ is odd.

6. Suppose a is an integer. If $7|4a$, then $7|a$.

Solution

$4a = 7c$ for some integer c.

Since $4a = 2(2a)$ it follows that $4a$ is even, and since $4a = 7c$, we know $7c$ is even. But then c cannot be odd, because that would make $7c$ odd, not even. Thus c is even, so $c = 2d$ for some integer d.

Now go back to the equation $4a = 7c$ and plug in $c = 2d$. We get $4a = 14d$. Dividing both sides by 2 gives $2a = 7d$. Now, since $2a = 7d$, it follows that $7d$ is even, and thus d cannot be odd. Then d is even, so $d = 2e$ for some integer e.

Plugging $d = 2e$ back into $2a = 7d$ gives $2a = 14e$. Dividing both sides of $2a = 14e$ by 2 produces $a = 7e$.

Finally, the equation $a = 7e$ means that $7|a$.

7. If a, b, and $c \in \mathbb{N}$, and $c \leqslant b \leqslant a$, then

$$\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}.$$

Solution

$$\begin{aligned}\binom{a}{b}\binom{b}{c} &= \frac{a!}{b!(a-b)!} \cdot \frac{b!}{c!(b-c)!} \\ &= \frac{a!}{(a-b)!} \cdot \frac{1}{c!(b-c)!} \\ &= \frac{a!}{(a-b+c)!(a-b)!} \cdot \frac{(a-b+c)!}{c!(b-c)!} \\ &= \frac{a!}{(b-c)![a-(b-c)]!} \cdot \frac{[(a-b)+c]!}{(a-b)!c!} \\ &= \binom{a}{b-c}\binom{a-b+c}{c},\end{aligned}$$

as required.

8. Prove that the sum of three consecutive numbers is a multiple of 3.

Solution

Three consecutive numbers are n, $(n+1)$, and $(n+2)$. Now, the sum of these three numbers is

$$\begin{aligned} n + (n+1) + (n+2) &= 3n + 3 \\ &= 3(n+1), \end{aligned}$$

and so the sum of three consecutive numbers is a multiple of 3.

9. Prove that the sum of two consecutive odd numbers is a multiple of 4.

Solution

Two consecutive odd numbers are $(2n+1)$ and $(2n+3)$. Now, the sum of these two consecutive odd numbers are

$$\begin{aligned} (2n+1) + (2n+3) &= 4n + 4 \\ &= 4(n+1), \end{aligned}$$

and so two consecutive odd numbers is a multiple of 4.

10. Denise has a multiple of 8. She adds 3 to this number and then squares the number. Prove that the resulting number is odd.

Solution

She begin with $8n + 3$ where $n \in \mathbb{N}$. Now,

$$\begin{aligned} (8n+3)^2 &= 64n^2 + 48n + 9 \\ &= 2(32n^2 + 24n + 4) + 1, \end{aligned}$$

which is always odd because it is of the form $2m + 1$ for some integer m.

11. (a) Find an algebraic expression for the difference between the squares of any two consecutive numbers.

Solution

We choose n and $(n+1)$ where $n \in \mathbb{N}$. So

$$\underline{\underline{(n+1)^2 - n^2}}.$$

(b) Hence, prove that the difference between the squares of any two consecutive numbers leaves a remainder of 1 when divided by 2.

Solution

$$\begin{aligned}(n+1)^2 - n^2 &= (n^2 + 2n + 1) - n^2 \\ &= 2n + 1,\end{aligned}$$

and which leave a remainder of 1 when divided by 2.