Dr Oliver Mathematics
Mathematics: Advanced Higher
2010 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Differentiate the following functions.

(a) $f(x) = e^x \sin x^2$. (3)

Solution

$$u = e^x \Rightarrow \frac{du}{dx} = e^x$$
$$v = \sin x^2 \Rightarrow \frac{dv}{dx} = 2x \cos x^2.$$

Finally,

$$\begin{aligned} f'(x) &= (e^x) \cdot (2x \cos x^2) + (\sin x^2) \cdot (e^x) \\ &= \underline{\underline{e^x(\sin x^2 + 2x \cos x^2)}}. \end{aligned}$$

(b) $g(x) = \dfrac{x^3}{1 + \tan x}$. (3)

Solution

$$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$$
$$v = 1 + \tan x \Rightarrow \frac{dv}{dx} = \sec^2 x.$$

Finally,

$$\begin{aligned} g'(x) &= \frac{(1 + \tan x) \cdot (3x^2) - (x^3) \cdot (\sec^2 x)}{(1 + \tan x)^2} \\ &= \underline{\underline{\frac{x^2[3(1 + \tan x) - x \sec^2 x]}{(1 + \tan x)^2}}}. \end{aligned}$$

2. The second and third terms of a geometric series are -6 and 3 respectively. Explain why the series has a sum to infinity, and obtain this sum. (5)

Solution

$$r = \frac{3}{-6} = -\tfrac{1}{2}$$

and so the sum to infinity has a limit as $|r| < 1$. Now,

$$a = \frac{ar}{r} = \frac{-6}{-\frac{1}{2}} = 12$$

and

$$\begin{aligned} S_\infty &= \frac{12}{1 - (-\frac{1}{2})} \\ &= \frac{12}{\frac{3}{2}} \\ &= \underline{\underline{8}}. \end{aligned}$$

3. (a) Use the substitution $t = x^4$ to obtain (3)

$$\int \frac{x^3}{1 + x^8}\,\mathrm{d}x.$$

Solution

$$\begin{aligned} t = x^4 &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = 4x^3 \\ &\Rightarrow \mathrm{d}t = 4x^3\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int \frac{x^3}{1 + x^8}\,\mathrm{d}x &= \tfrac{1}{4}\int \frac{4x^3}{1 + (x^4)^2}\,\mathrm{d}x \\ &= \tfrac{1}{4}\int \frac{1}{1 + u^2}\,\mathrm{d}u \\ &= \tfrac{1}{4}\arctan u + c \\ &= \underline{\underline{\tfrac{1}{4}\arctan x^4 + c}}. \end{aligned}$$

(b) Integrate $x^2 \ln x$ with respect to x. (4)

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\mathrm{d}v = x^2 \Rightarrow v = \tfrac{1}{3}x^3.$$

$$\begin{aligned}\int x^2 \ln x \,\mathrm{d}x &= \tfrac{1}{3}x^3 \ln x - \int \left(\tfrac{1}{3}x^3 \cdot \frac{1}{x}\right) \mathrm{d}x \\ &= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2 \,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + c}}.\end{aligned}$$

4. Obtain the 2×2 matrix $\mathbf{M}$ associated with an enlargement, scale factor 2, followed by a clockwise rotation of 60° about the origin. (4)

Solution

$$\begin{aligned}\mathbf{M} &= \begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}}}.\end{aligned}$$

5. Show that (4)

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2},$$

where the integer n is greater than or equal to 3.

Solution

$$\begin{aligned}
\binom{n+1}{3} - \binom{n}{3} &= \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!} \\
&= \frac{n!}{3!(n-3)!}\left[\frac{n+1}{n-2} - 1\right] \\
&= \frac{n!}{3!(n-3)!}\left[\frac{(n+1)-(n-2)}{n-2}\right] \\
&= \frac{n!}{3!(n-3)!}\left[\frac{3}{n-2}\right] \\
&= \frac{n!}{2!(n-2)!} \\
&= \underline{\underline{\binom{n}{2}}},
\end{aligned}$$

as required.

6. Given (4)

$$\mathbf{u} = -2\mathbf{i} + 5\mathbf{k},\ \mathbf{v} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k},\ \text{ and } \mathbf{w} = -\mathbf{i}+\mathbf{j}+4\mathbf{k},$$

calculate

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}).$$

Solution

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} \\
&= 9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u}.(\mathbf{v} \times \mathbf{w}) &= (-2\mathbf{i} + 5\mathbf{k}).(9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}) \\
&= -18 + 0 + 25 \\
&= \underline{\underline{7}}.
\end{aligned}$$

7. Evaluate (6)

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,\mathrm{d}x,$$

expressing your answer in the form $\ln \frac{a}{b}$, where a and b are integers.

Solution

$$\begin{aligned}\frac{3x+5}{(x+1)(x+2)(x+3)} &\equiv \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)} \\ &\equiv \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}\end{aligned}$$

and so

$$3x+5 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2).$$

$\underline{x=-1}$: $2 = 2A \Rightarrow A = 1$.
$\underline{x=-2}$: $-1 = -B \Rightarrow B = 1$.
$\underline{x=-3}$: $-4 = 2C \Rightarrow C = -2$.
Finally,

$$\begin{aligned}\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,\mathrm{d}x &= \int_1^2 \left(\frac{1}{(x+1)} + \frac{1}{(x+2)} - \frac{2}{(x+3)}\right) \mathrm{d}x \\ &= \left[\ln|x+1| + \ln|x+2| - 2\ln|x+3|\right]_{x=1}^2 \\ &= (\ln 3 + \ln 4 - 2\ln 5) - (\ln 2 + \ln 3 - 2\ln 4) \\ &= \ln \tfrac{12}{25} - \ln \tfrac{3}{8} \\ &= \underline{\underline{\ln \tfrac{32}{25}}};\end{aligned}$$

hence, $\underline{\underline{a = 32}}$ and $\underline{\underline{b = 25}}$.

8. (a) Prove that the product of two odd integers is odd. (2)

Solution
Let $2m+1$ and $2n+1$ be two odd integers, where $m, n \in \mathbb{Z}$. Then

$$\begin{aligned}(2m+1)(2n+1) &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2 \times \text{some integer} + 1;\end{aligned}$$

hence, the product of two odd integers is $\underline{\underline{\text{odd}}}$.

(b) Let p be an odd integer. Use the result of (a) to prove by induction that p^n is odd for all positive integers n. (4)

Solution

$\underline{n=1}$: $p^1 = p =$ odd integer and so the case $n = 1$ is true.
Suppose it is true for $n = k$, i.e., p^k is odd. Then

$$\begin{aligned} p^{k+1} &= p^k p \\ &= \text{odd integer} \times \text{odd integer} \\ &= \text{odd integer} \end{aligned}$$

and we have proved p^{k+1}.
Hence, by mathematical induction, p^n is $\underline{\underline{\text{odd}}}$ for all positive integers n.

9. Obtain the first three non-zero terms in the Maclaurin expansion of (4)

$$1 + \sin^2 x.$$

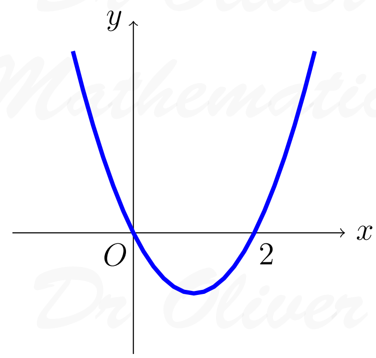
Solution

	Function	Function evaluated at 0
Original	$1 + \sin^2 x$	1
First derivative	$2\sin x \cos x = \sin 2x$	0
Second derivative	$2\cos 2x$	2
Third derivative	$-4\sin 2x$	0
Fourth derivative	$-8\cos 2x$	-8

Hence,

$$\begin{aligned} 1 + \sin^2 x &= 1 + \tfrac{1}{2!}(2)x^2 + \tfrac{1}{4!}(-8)x^4 + \ldots \\ &= \underline{\underline{1 + x^2 + \tfrac{1}{3}x^4 + \ldots}}. \end{aligned}$$

10. The diagram below shows part of the graph of a function $\mathrm{f}(x)$. (3)



State whether $\mathrm{f}(x)$ is odd, even, or neither.
Fully justify your answer.

Solution

$\underline{\mathrm{f}(x) = \mathrm{f}(-x)}$? No: $\mathrm{f}(2) = 0$ but $\mathrm{f}(-2) > 0$.
$\underline{\mathrm{f}(-x) = -\mathrm{f}(x)}$? No: $\mathrm{f}(-2) > 0$ but $-\mathrm{f}(2) = 0$.
Hence, $\mathrm{f}(x)$ is neither even or odd.

11. (a) Obtain the general solution of the equation (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

Solution

Complementary function:

$$\begin{aligned} m^2 + 4m + 5 = 0 &\Rightarrow m^2 + 4m + 4 = -1 \\ &\Rightarrow (m+2)^2 = -1 \\ &\Rightarrow m + 2 = \pm \mathrm{i} \\ &\Rightarrow m = -2 \pm \mathrm{i} \end{aligned}$$

and the complementary function is

$$\underline{\underline{y = \mathrm{e}^{-2x}(A\cos x + B\sin x)}}.$$

(b) Hence, obtain the solution for which $y = 3$ when $x = 0$ and $y = \mathrm{e}^{-\pi}$ when $x = \frac{1}{2}\pi$. (3)

Solution

$$x = 0, y = 3 \Rightarrow 3 = A$$

and

$$x = \tfrac{1}{2}\pi, y = \mathrm{e}^{-\pi} \Rightarrow \mathrm{e}^{-\pi} = B\mathrm{e}^{-\pi} \Rightarrow B = 1.$$

Finally,

$$\underline{\underline{y = \mathrm{e}^{-2x}(3\cos x + \sin x)}}.$$

12. Prove by contradiction that, if x is an irrational number, then $2 + x$ is irrational. (4)

Solution

Suppose that $2 + x$ is rational:

$$2 + x = \frac{p}{q},$$

where p and q are integers. Then

$$\begin{aligned} x &= (2 + x) - 2 \\ &= \frac{p}{q} - 2 \\ &= \frac{p - 2q}{q}; \end{aligned}$$

now, $(p - 2q)$ and q are integers, which is a contradiction.
Hence, if x is an irrational number, then $2 + x$ is irrational.

13. (a) Given (4)

$$y = t^3 - \tfrac{5}{2}t^2 \text{ and } x = \sqrt{t} \text{ for } t > 0,$$

use parametric differentiation to express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t in simplified form.

Solution

$$\begin{aligned} y = t^3 - \tfrac{5}{2}t^2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 5t \\ x = \sqrt{t} &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{1}{2}t^{-\frac{1}{2}}. \end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{3t^2 - 5t}{\frac{1}{2}t^{-\frac{1}{2}}} \\ &= \underline{\underline{6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}}}.\end{aligned}$$

(b) Show that (3)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = at^2 + bt,$$

determining the values of the constants a and b.

Solution

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \left(15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}\right) \cdot 2t^{\frac{1}{2}} \\ &= \underline{\underline{30t^2 - 30t}};\end{aligned}$$

hence, $\underline{\underline{a = 30}}$ and $\underline{\underline{b = -30}}$.

(c) Obtain an equation for the tangent to the curve which passes through the point of inflexion. (3)

Solution

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 &\Rightarrow 30t^2 - 30t = 0 \\ &\Rightarrow 30t(t-1) = 0 \\ &\Rightarrow t = 0 \text{ or } t = 1.\end{aligned}$$

Hence, $t = 1$ for this question (why?) and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1} = -4.$$

Now,

$$t = 1 \Rightarrow x = 1$$

and

$$t = 1 \Rightarrow y = -\tfrac{3}{2}.$$

Finally, an equation for the tangent to the curve is

$$\begin{aligned} y + \tfrac{3}{2} = -4(x-1) &\Rightarrow y + \tfrac{3}{2} = -4x + 4 \\ &\Rightarrow \underline{\underline{y = -4x + \tfrac{5}{2}}}. \end{aligned}$$

14. (a) Use Gaussian elimination to show that the set of equations (5)

$$\begin{aligned} x - y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y + az &= 2 \end{aligned}$$

has a unique solution when $a \neq 2.5$.

Solution

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right)$$

Do $R_2 - R_1$ and $R_3 - 2 \times R_1$:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right)$$

Do $R_3 - \frac{1}{2} \times R_2$:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & a-\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

Now, the solutions:

$$(a - \tfrac{1}{2})z = \tfrac{1}{2} \Rightarrow z = \frac{1}{2a-5},$$

$$2y + \frac{1}{2a-5} = -1 \Rightarrow 2y = -1 - \frac{1}{2a-5}$$
$$\Rightarrow 2y = \frac{-(2a-5)-1}{2a-5}$$
$$\Rightarrow 2y = \frac{4-2a}{2a-5}$$
$$\Rightarrow y = \frac{2-a}{2a-5},$$

and

$$x - \frac{2-a}{2a-5} + \frac{1}{2a-5} = 1 \Rightarrow x + \frac{a-1}{2a-5} = 1$$
$$\Rightarrow x = 1 - \frac{a-1}{2a-5}$$
$$\Rightarrow x = \frac{(2a-5)-(a-1)}{2a-5}$$
$$\Rightarrow x = \frac{a-4}{2a-5}.$$

Hence,

$$\underline{\underline{x = \frac{a-4}{2a-5}, y = \frac{2-a}{2a-5}, z = \frac{1}{2a-5}}}.$$

(b) Explain what happens when $a = 2.5$. (1)

Solution
From the third row of the final tableau, when $a = 2.5$, there are <u>no solutions</u>.

(c) Obtain the solution when $a = 3$. (1)

Solution

$$a = 3 \Rightarrow \underline{\underline{x = -1, y = -1, z = 1}}.$$

(d) Given (1)

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

calculate **AB**.

Solution

$$\mathbf{AB} = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}}.$$

(e) Hence, or otherwise, state the relationship between **A** and the matrix (2)

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}.$$

Solution

$$\mathbf{AC} = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{CA} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which means they are inverses of each other:

$$\underline{\underline{\mathbf{A} = \mathbf{C}^{-1}}}.$$

15. (a) A new board game has been invented and the symmetrical design on the board is made from four identical "petal" shapes. One of these petals is the region enclosed (5)

between the curves

$$y = x^2 \text{ and } y^2 = 8x,$$

as shown shaded in diagram 1 below.

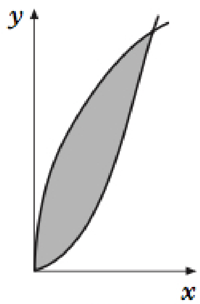


Diagram 1

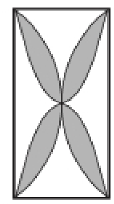


Diagram 2

Calculate the area of the complete design, as shown in diagram 2.

Solution

Where do the curves meet?

$$\begin{aligned} x^2 = \sqrt{8x} &\Rightarrow x^2 - \sqrt{8x} = 0 \\ &\Rightarrow x^{\frac{1}{2}}(x^{\frac{3}{2}} - \sqrt{8}) = 0 \\ &\Rightarrow x = 0 \text{ or } x^{\frac{3}{2}} = \sqrt{8} \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area} &= 4 \int_0^2 (\sqrt{8x} - x^2)\,\mathrm{d}x \\ &= 4 \left[\sqrt{8} \cdot \tfrac{2}{3} x^{\frac{3}{2}} - \tfrac{1}{3} x^3 \right]_{x=0}^{2} \\ &= 4 \left[\left(5\tfrac{1}{3} - 2\tfrac{2}{3}\right) - (0 - 0) \right] \\ &= \underline{\underline{10\tfrac{2}{3}}}. \end{aligned}$$

(b) The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through 360° about the y-axis. Find the volume of plastic required to make one counter. (5)

Solution

When $x = 2$, $y = 4$ and we need

$$\text{volume} = \pi \int x^2 \, \mathrm{d}y.$$

Now,

$$\begin{aligned}
\text{volume} &= \pi \int_0^4 y \, \mathrm{d}y - \pi \int_0^4 \left(\frac{y^2}{8} \right)^2 \mathrm{d}y \\
&= \pi \int_0^4 \left(y - \frac{1}{64} y^4 \right) \mathrm{d}y \\
&= \pi \left[\tfrac{1}{2} y^2 - \tfrac{1}{320} y^5 \right]_{y=0}^4 \\
&= \pi \left[\left(8 - 3\tfrac{1}{5} \right) - (0 - 0) \right] \\
&= \underline{\underline{\tfrac{24}{5}\pi}}.
\end{aligned}$$

16. (a) Given (1)

$$z = r(\cos\theta + \mathrm{i}\sin\theta),$$

use de Moivre's theorem to express z^3 in polar form.

Solution

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \Rightarrow \underline{\underline{z^3 = r^3(\cos 3\theta + \mathrm{i}\sin 3\theta)}}.$$

(b) Hence obtain (2)

$$\left(\cos\tfrac{2}{3}\pi + \mathrm{i}\sin\tfrac{2}{3}\pi \right)^3$$

in the form $a + \mathrm{i}b$.

Solution

$$\begin{aligned}
\left(\cos\tfrac{2}{3}\pi + \mathrm{i}\sin\tfrac{2}{3}\pi \right)^3 &= \cos 2\pi + \mathrm{i}\sin 2\pi \\
&= \underline{\underline{1}};
\end{aligned}$$

hence, $\underline{\underline{a = 1}}$ and $\underline{\underline{b = 0}}$.

(c) Hence, or otherwise, obtain the roots of the equation (4)

$$z^3 = 8$$

in Cartesian form.

Solution

$$\begin{aligned} z^3 = 8 \Rightarrow z^3 &= 8(\cos 2n\pi + \mathrm{i} \sin 2n\pi) \text{ for some scalar } n \\ \Rightarrow z^3 &= 2(\cos \tfrac{2n\pi}{3} + \mathrm{i} \sin \tfrac{2n\pi}{3}); \end{aligned}$$

Now,

$$\begin{aligned} n = 0 &\Rightarrow \underline{\underline{z = 2}} \\ n = 1 &\Rightarrow z = 2(\cos \tfrac{2\pi}{3} + \mathrm{i} \sin \tfrac{2\pi}{3}) = \underline{\underline{-1 + \mathrm{i}\sqrt{3}}} \\ n = 2 &\Rightarrow z = 2(\cos \tfrac{4\pi}{3} + \mathrm{i} \sin \tfrac{4\pi}{3}) = \underline{\underline{-1 - \mathrm{i}\sqrt{3}}}. \end{aligned}$$

(d) Denoting the roots of (3)

$$z^3 = 8$$

by z_1, z_2, and z_3:

(i) state the value $z_1 + z_2 + z_3$;

Solution

$$z_1 + z_2 + z_3 = \underline{\underline{0}}$$

as there is no z^2 term.

(ii) obtain the value of $z_1^6 + z_2^6 + z_3^6$.

Solution

$$\begin{aligned} z_1^6 + z_2^6 + z_3^6 &= (z_1^3)^2 + (z_2^3)^2 + (z_3^3)^2 \\ &= 8^2 + 8^2 + 8^2 \\ &= \underline{\underline{192}}. \end{aligned}$$