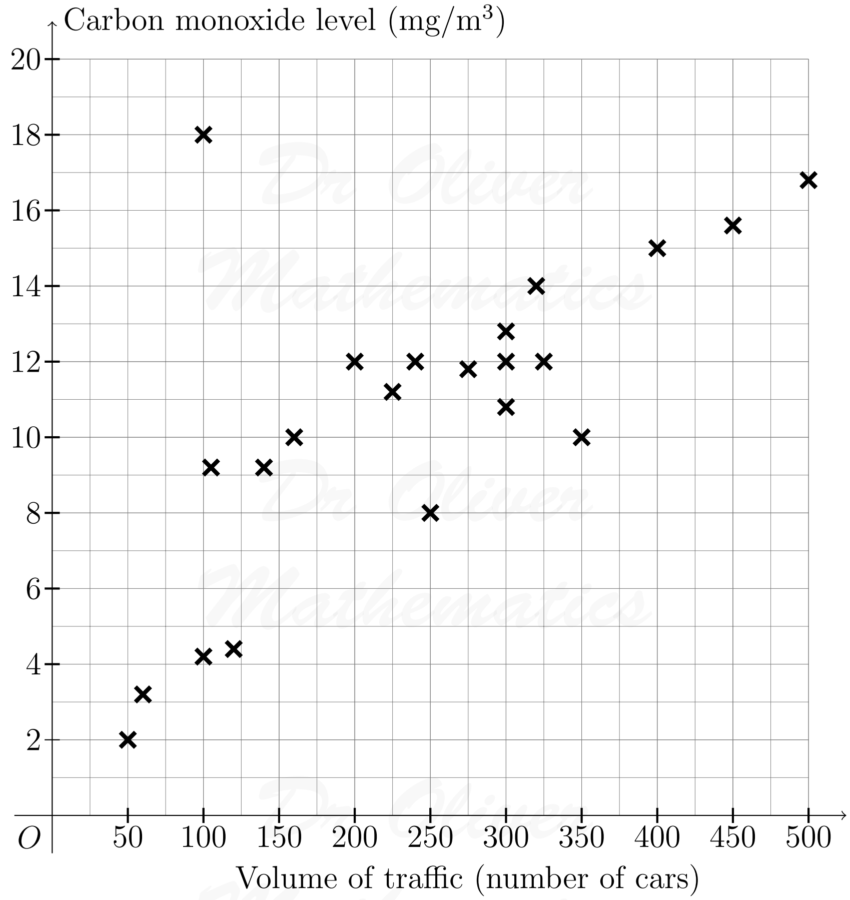
Dr Oliver Mathematics
GCSE Mathematics
2021 November Paper 3H: Calculator
1 hour 30 minutes

The total number of marks available is 80.
You must write down all the stages in your working.

1. The scatter graph shows information about the volume of traffic and the carbon monoxide level at a point on a road each day for 22 days.



One point is an outlier.

(a) Write down the coordinates of this point. (1)

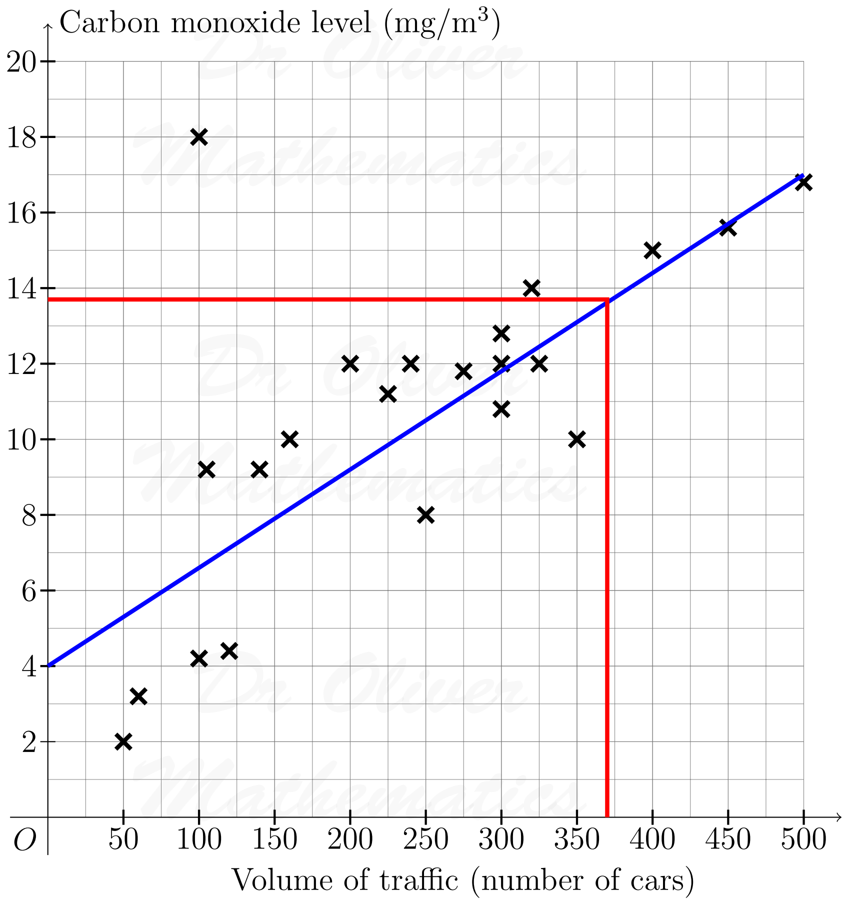
Solution
$\underline{\underline{(100, 18)}}$.

For another day, 370 cars pass the point on the road.

(b) Estimate the carbon monoxide level for this day. (2)

Solution

Draw a line of best fit:



Correct read-off: approximately 13.7.

Alfie says, "Because there is an outlier, there is no correlation."

(c) Is Alfie correct? (1)

You must give a reason for your answer.

Solution

No: because even with an outlier you can still have a negative or positive correlation; as a line of best fit can still be drawn.

2. Natalie makes potato cakes in a restaurant. (4)

She mixes potato, cheese and onion so that

$$\text{weight of potato} : \text{weight of cheese} : \text{weight of onion} = 9 : 2 : 1.$$

Natalie needs to make 6 000 g of potato cakes.

Cheese costs £2.25 for 175 g.

Work out the cost of the cheese needed to make 6 000 g of potato cakes.

Solution

Well,

$$9 + 2 + 1 = 12$$

so the 'weight' of cheese is

$$\tfrac{2}{12} \times 6\,000 = 1\,000 \text{ g}.$$

Hence, the cost of the cheese is

$$\frac{1\,000}{175} \times 2.25 = 12.\dot{8}57\,14\dot{2};$$

so, the cost of the cheese is

$$\underline{\underline{\pounds 12.86 \text{ (nearest penny)}}}.$$

3. (a) Write (1)

$$4.5 \times 10^5$$

as an ordinary number.

Solution

$$4.5 \times 10^5 = \underline{\underline{450\,000}}.$$

(b) Write (1)

$$0.007$$

in standard form.

Solution

$$0.007 = \underline{\underline{7 \times 10^{-3}}}.$$

(c) Work out (2)

$$4.2 \times 10^3 + 5.3 \times 10^2.$$

Give your answer in standard form.

Solution

$$\begin{aligned} 4.2 \times 10^3 + 5.3 \times 10^2 &\Rightarrow 4.2 \times 10^3 + 0.53 \times 10^3 \\ &\Rightarrow \underline{\underline{4.73 \times 10^3}}. \end{aligned}$$

4. A water tank is empty. (4)
Anil needs to fill the tank with 2 400 litres of water.

Company **A** supplies water at a rate of 8 litres in 1 minute 40 seconds.
Company **B** supplies water at a rate of 2.2 gallons per minute.

1 gallon = 4.54 litres.

Company **A** would take more time to fill the tank than Company **B** would take to fill the tank.

How much more time?
Give your answer in minutes correct to the nearest minute.

Solution

Company **A**:

$$\begin{aligned} & 8 \text{ litres} \leftrightarrow 1 \text{ minute } 40 \text{ seconds} \\ \Rightarrow \quad & 8 \text{ litres} \leftrightarrow \tfrac{5}{3} \text{ minutes} \\ \Rightarrow \quad & 1 \text{ litre} \leftrightarrow \tfrac{5}{24} \text{ minutes} \\ \Rightarrow \quad & 2\,400 \text{ litres} \leftrightarrow 500 \text{ minutes}. \end{aligned}$$

Company **B**:

$$\begin{aligned} & 2.2 \text{ gallons} \leftrightarrow 1 \text{ minute} \\ \Rightarrow\ & 9.988 \text{ litres} \leftrightarrow 1 \text{ minute} \\ \Rightarrow\ & 1 \text{ litre} \leftrightarrow \tfrac{250}{2\,497} \text{ minutes} \\ \Rightarrow\ & 2\,400 \text{ litres} \leftrightarrow 240.288\,346 \text{ minutes (FCD)}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{time} &= 500 - 240.288\,346 \text{ mins (FCD)} \\ &= 259.711\,654 \text{ mins (FCD)} \\ &= \underline{\underline{260 \text{ mins (nearest minute)}}}. \end{aligned}$$

5. The first four terms of a Fibonacci sequence are (3)

$$a \qquad 2a \qquad 3a \qquad 5a.$$

The sum of the first five terms of this sequence is 228.

Work out the value of a.

Solution

Clearly, the next term in the sequence is

$$3a + 5a = 8a.$$

Now,

$$\begin{aligned} a + 2a + 3a + 5a + 8a = 228 &\Rightarrow 19a = 228 \\ &\Rightarrow \underline{\underline{a = 12}}. \end{aligned}$$

6. In a bag there are only red counters, blue counters, green counters, and pink counters. A counter is going to be taken at random from the bag.

The table shows the probabilities of taking a red counter or a blue counter.

Colour	Red	Blue	Green	Pink
Probability	0.05	0.15		

The probability of taking a green counter is 0.2 more than the probability of taking a pink counter.

(a) Complete the table. (2)

Solution

Well,

$$0.05 + 0.15 = 0.2$$

so the probabilities of either green or pink add up to 0.8:

$$\mathrm{P}(G) + \mathrm{P}(P) = 0.8 \quad (1)$$

and their difference is 0.2:

$$\mathrm{P}(G) - \mathrm{P}(P) = 0.2 \quad (2).$$

Add (1) + (2):

$$2\,\mathrm{P}(G) = 1 \Rightarrow \mathrm{P}(G) = 0.5$$
$$\Rightarrow \mathrm{P}(P) = 0.3.$$

Now, can we complete the table:

Colour	Red	Blue	Green	Pink
Probability	0.05	0.15	$\underline{\underline{0.5}}$	$\underline{\underline{0.3}}$

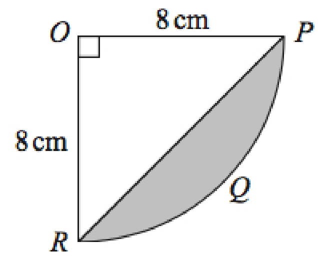
There are 18 blue counters in the bag.

(b) Work out the total number of counters in the bag. (2)

Solution

$$\text{Total number} = \frac{18}{0.15} = \underline{\underline{120}}.$$

7. The diagram shows a sector $OPQR$ of a circle, centre O and radius 8 cm. (4)



OPR is a triangle.

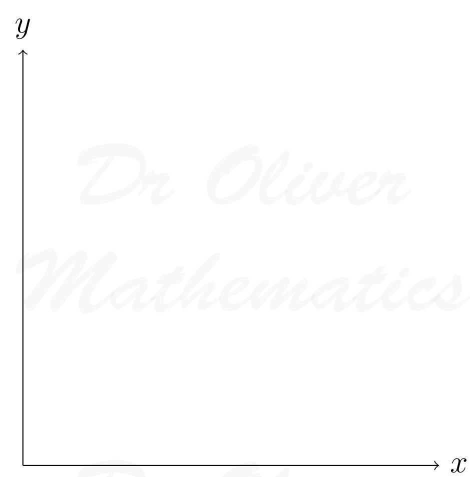
Work out the area of the shaded segment PQR.
Give your answer correct to 3 significant figures.

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{4}(\pi \times 8^2) - \tfrac{1}{2} \times 8 \times 8) \\ &= 16\pi - 32 \text{ (exact!)} \\ &= \underline{\underline{18.3 \text{ cm}^2 \text{ (3 sf)}}}. \end{aligned}$$

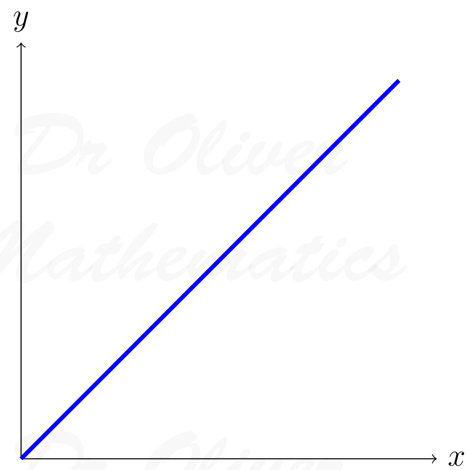
8. (a) Using the axes below, sketch a graph to represent the statement (1)

y is directly proportional to x.



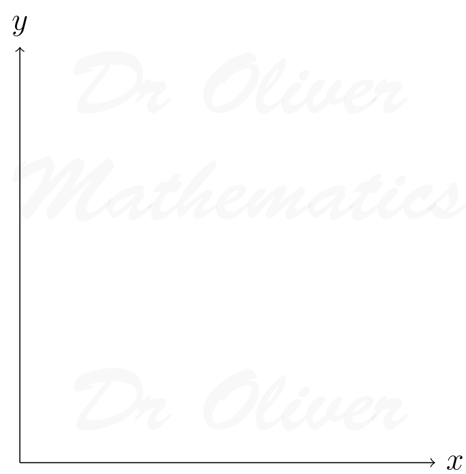
Solution

E.g.,



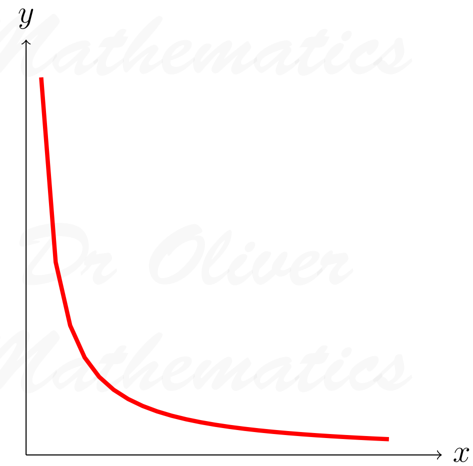
(b) Using the axes below, sketch a graph to represent the statement (1)

y is inversely proportional to x.



Solution

E.g.,



9. On Monday, 12 people took 5 hours to clean a number of cars.
On Tuesday, 15 people cleaned the same number of cars.

Assuming that all the people worked at the same rate,

(a) work out how many hours the 15 people took to clean the cars. (2)

Solution

On Monday,

$$12 \times 5 = 60$$

and, on Tuesday,

$$\begin{aligned} \text{clean the cars} &= \frac{60}{15} \\ &= \underline{\underline{4 \text{ hours}}}. \end{aligned}$$

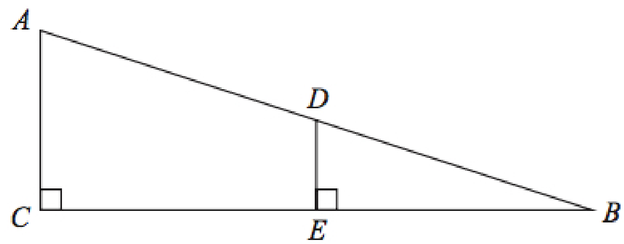
The assumption is wrong.

(b) How might this affect the time taken for the 15 people to clean the cars? (1)

Solution

E.g., some workers would be <u>slower</u> than others.

10. The diagram shows two right-angled triangles ACB and DEB. (4)



- $AD = 9$ cm.
- $DE = 2$ cm.
- $DB = 6$ cm.

Calculate the length of CB.
Give your answer correct to 2 decimal places.

Solution

Now,

$$\begin{aligned}\frac{AC}{DE} = \frac{AB}{DB} &\Rightarrow \frac{AC}{2} = \frac{9+6}{6} \\ &\Rightarrow \frac{AC}{2} = \tfrac{5}{62} \\ &\Rightarrow AC = \tfrac{5}{2} \times 2 \\ &\Rightarrow AC = 5 \text{ cm}\end{aligned}$$

and

$$\begin{aligned}AC^2 + CB^2 = AB^2 &\Rightarrow 5^2 + CB^2 = 15^2 \\ &\Rightarrow 25 + CB^2 = 225 \\ &\Rightarrow CB^2 = 200 \\ &\Rightarrow CB = 14.142\,135\,62 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{CB = 14.14 \text{ cm (2 dp)}}}.\end{aligned}$$

11. Freya writes down the value of x, correct to 1 decimal place. (2)

She writes $x = 6.4$.

Complete the error interval for x.

Solution

$$\underline{\underline{6.35 \leqslant x < 6.45}}.$$

12. (2)

$$(ax^6)^{\frac{1}{n}} = 7x^3.$$

Work out the value of a and the value of n.

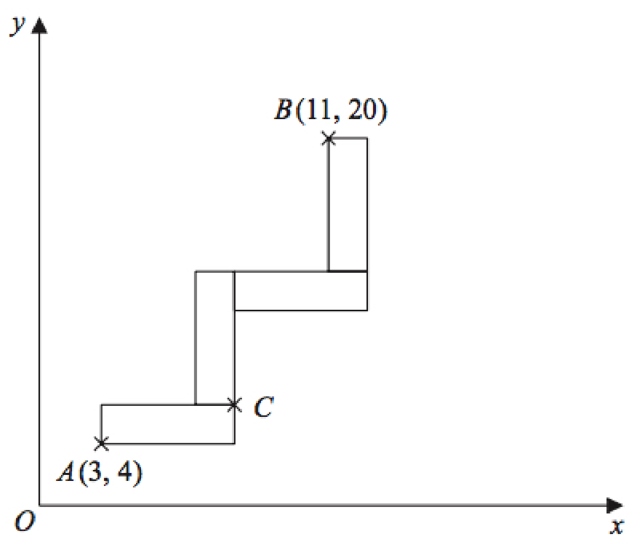
Solution

$$\begin{aligned}(ax^6)^{\frac{1}{n}} = 7x^3 &\Rightarrow ax^6 = (7x^3)^n \\ &\Rightarrow ax^6 = 7^n x^{3n} \\ &\Rightarrow ax^{6-3n} = 7^n;\end{aligned}$$

hence,

$$\underline{\underline{n = 2 \text{ and } a = 49}}.$$

13. A pattern is made from four identical rectangles. (5)
The sides of the rectangles are parallel to the axes.



Point A has coordinates $(3, 4)$.
Point B has coordinates $(11, 20)$.

Point C is marked on the diagram.

Work out the coordinates of C.
You must show all your working.

Solution

Let

$$\overrightarrow{AC} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

For the x-coordinate,

$$3 + c + c - d = 11 \Rightarrow 2c - d = 8 \quad (1)$$

and, the y-coordinate,

$$4 + d + c + c = 20 \Rightarrow 2c + d = 16 \quad (2).$$

Add (1) + (2):

$$\begin{aligned} 4c = 24 &\Rightarrow c = 6 \\ &\Rightarrow d = 4; \end{aligned}$$

hence,

$$(3 + 6, 4 + 4) = \underline{\underline{C(9, 8)}}.$$

14. Olivia and Jessica have in total half as many sweets as Fran and Gary have in total. (4)

Fran and Gary share their sweets in the ratio 2 : 3.
Olivia and Jessica share their sweets in the ratio 9 : 1.

Fran got w sweets.
Gary got x sweets.
Olivia got y sweets.
Jessica got z sweets.

Find, in its simplest form,

$$w : x : y : z.$$

Solution

Let Olivia and Jessica have s sweets. Then Fran and Gary have $2s$ sweets. Now,

$$2 + 3 = 5$$

and

$$\begin{aligned} \text{Fran:} \quad & \tfrac{2}{5} \times 2s = \tfrac{4}{5}s \\ \text{Gary:} \quad & \tfrac{3}{5} \times 2s = \tfrac{6}{5}s. \end{aligned}$$

Next,

$$9 + 1 = 10$$

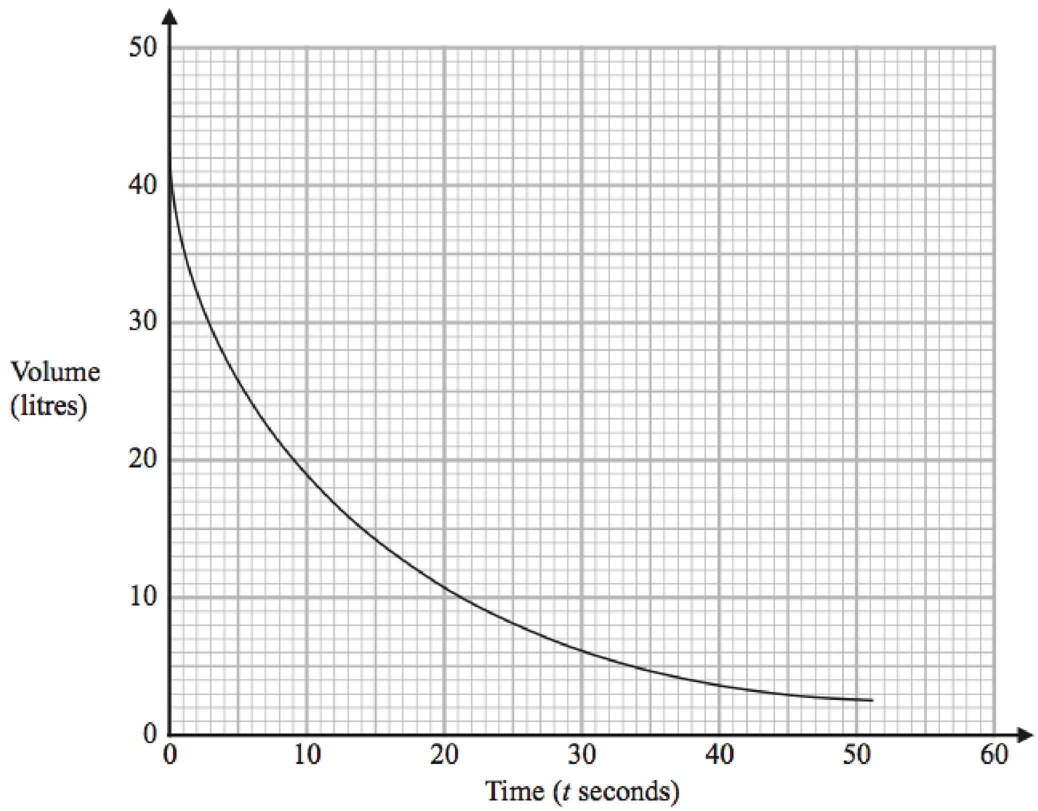
and

$$\begin{aligned} \text{Olivia:} \quad & \tfrac{9}{10} \times s = \tfrac{9}{10}s \\ \text{Jessica:} \quad & \tfrac{1}{10} \times s = \tfrac{1}{10}s. \end{aligned}$$

Hence,

$$\begin{aligned} w : x : y : z &= \tfrac{4}{5}s : \tfrac{6}{5}s : \tfrac{9}{10}s : \tfrac{1}{10}s \\ &= \tfrac{8}{10} : \tfrac{12}{10} : \tfrac{9}{10} : \tfrac{1}{10} \\ &= \underline{\underline{8 : 12 : 9 : 1}}. \end{aligned}$$

15. The graph gives the volume of water, in litres, in a container at time t seconds after the water started to flow out of the container. (3)



Using the graph, work out an estimate for the rate at which the water is flowing out of the container when $t = 12$.
You must show your working.

Solution

Draw a tangent to the curve: it will go through the points $(28, 0)$ and $(0, 28)$. Hence, an estimate for the rate at which the water is flowing out of the container is 1 litre/s.

16. The curve C has equation (4)

$$y = x^2 + 3x - 3.$$

The line **L** has equation

$$y - 5x + 4 = 0.$$

Show, algebraically, that C and **L** have exactly one point in common.

Solution

Well,

$$y - 5x + 4 = 0 \Rightarrow y = 5x - 4.$$

Simultaneous equations:

$$x^2 + 3x - 3 = 5x - 4 \Rightarrow x^2 - 2x + 1 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -2 \\ \text{multiply to:} & +1\end{array}\right\} -1,\ -1$$

$$\Rightarrow (x-1)^2 = 0$$
$$\Rightarrow x = 1 \text{ (repeated)};$$

hence, $\underline{\underline{(1, 1)}}$ is the <u>only</u> point that they have in common.

17. x is directly proportional to the square of y. (4)
y is directly proportional to the cube of z.

$z = 2$ when $x = 32$.

Find a formula for x in terms of z.

Solution

Well,

$$x \propto y^2 \Rightarrow x = ky^2$$

and

$$y \propto z^3 \Rightarrow y = lz^3,$$

for some k and l. Then,

$$\begin{aligned} x &= ky^2 \\ &= k(lz^3)^2 \\ &= cz^6, \end{aligned}$$

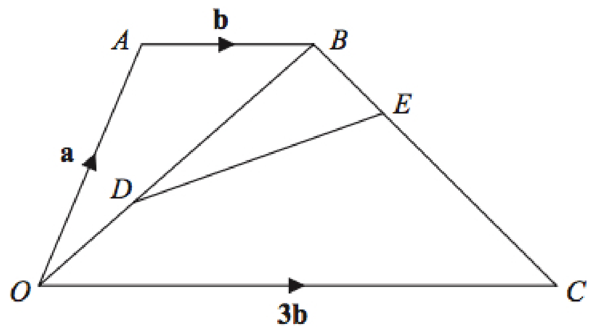
where $c = kl^6$. Now,

$$32 = c \times 2^6 \Rightarrow c = \tfrac{1}{2}$$

and, finally,

$$\underline{\underline{x = \tfrac{1}{2}z^6}}.$$

18. $OABC$ is a trapezium. (4)



$\overrightarrow{OA} = \mathbf{a}$.
$\overrightarrow{AB} = \mathbf{b}$.
$\overrightarrow{OC} = 3\mathbf{b}$.

D is the point on OB such that $OD : DB = 2 : 3$.
E is the point on BC such that $BE : EC = 1 : 4$.

Work out the vector $\overrightarrow{DE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Give your answer in its simplest form.

Solution

Well,

$$\begin{aligned}\overrightarrow{DB} &= \tfrac{3}{5}\overrightarrow{OB} \\ &= \tfrac{3}{5}(\overrightarrow{OA} + \overrightarrow{AB}) \\ &= \tfrac{3}{5}(\mathbf{a} + \mathbf{b})\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{BE} &= \tfrac{1}{5}\overrightarrow{BC} \\ &= \tfrac{1}{5}(\overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC}) \\ &= \tfrac{1}{5}(-\overrightarrow{AB} - \overrightarrow{OA} + \overrightarrow{OC}) \\ &= \tfrac{1}{5}(-\mathbf{b} - \mathbf{a} + 3\mathbf{b}) \\ &= \tfrac{1}{5}(-\mathbf{a} + 2\mathbf{b}).\end{aligned}$$

Now,

$$\begin{aligned}\overrightarrow{DE} &= \overrightarrow{DE} + \overrightarrow{BE} \\ &= \tfrac{3}{5}(\mathbf{a}+\mathbf{b}) + \tfrac{1}{5}(-\mathbf{a}+2\mathbf{b}) \\ &= \tfrac{3}{5}\mathbf{a} + \tfrac{3}{5}\mathbf{b} - \tfrac{1}{5}\mathbf{a} + \tfrac{2}{5}\mathbf{b} \\ &= \underline{\underline{\tfrac{2}{5}\mathbf{a} + \mathbf{b}}}.\end{aligned}$$

19. At the start of year n, the number of animals in a population is P_n. (3)

At the start of the following year, the number of animals in the population is P_{n+1} where

$$P_{n+1} = kP_n.$$

At the start of 2017 the number of animals in the population was 4 000.
At the start of 2019 the number of animals in the population was 3 610.

Find the value of the constant k.

Solution

Well,

$$\begin{aligned}P_{n+2} &= kP_{n+1} \\ &= k(kP_n) \\ &= k^2 P_n.\end{aligned}$$

Now,

$$\begin{aligned}3\,610 = 4\,000k^2 \Rightarrow k^2 &= \tfrac{361}{400} \\ \Rightarrow \underline{\underline{k = \tfrac{19}{20}}}.\end{aligned}$$

20. Pat throws a fair coin n times. (2)

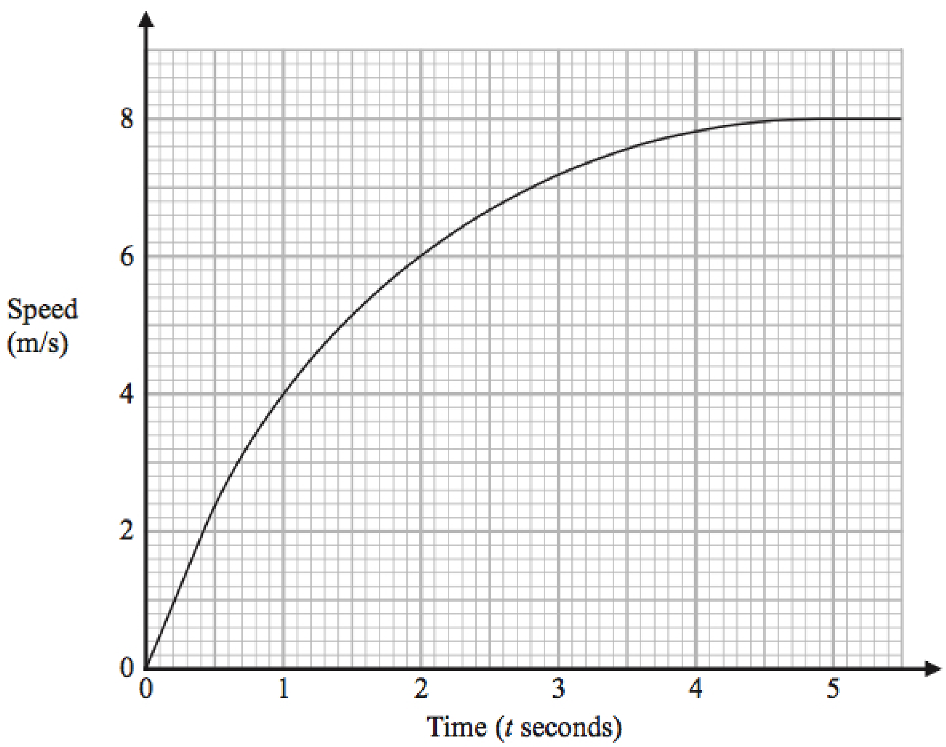
Find an expression, in terms of n, for the probability that Pat gets at least 1 head and at least 1 tail.

Solution

Well,

$$\begin{aligned} P(\text{least 1 head and at least 1 tail}) &= 1 - P(\text{no heads}) - P(\text{no tails}) \\ &= 1 - (\tfrac{1}{2})^n - (\tfrac{1}{2})^n \\ &= 1 - \tfrac{2}{2^n} \\ &= \underline{\underline{1 - \tfrac{1}{2^{n-1}}}}. \end{aligned}$$

21. Here is a speed-time graph showing the speed, in metres per second, of an object t seconds after it started to move from rest.



(a) Using 3 trapeziums of equal width, work out an estimate for the area under the graph between $t = 1$ and $t = 4$. (3)

Solution

Now,

$$
\begin{aligned}
t = 1 &\Rightarrow v = 4 \\
t = 2 &\Rightarrow v = 6 \\
t = 3 &\Rightarrow v = 7.2 \\
t = 4 &\Rightarrow v = 7.8.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\text{an estimate} &= \tfrac{1}{2} \times 1 \times [4 + 2(6 + 7.2) + 7.8] \\
&= \underline{\underline{19.1}}.
\end{aligned}
$$

(b) What does this area represent? (1)

Solution

It is an underestimate for the <u>distance</u> that the ball moves.

22. Show that (3)

$$\frac{6x^3}{(9x^2 - 144)} \div \frac{2x^4}{3(x-4)}$$

can be written in the form

$$\frac{1}{x(x+r)},$$

where r is an integer.

Solution

$$9x^2 - 144 = 9(x^2 - 16)$$

$$\left.\begin{array}{ll} \text{add to:} & 0 \\ \text{multiply to:} & -16 \end{array}\right\} + 4, -4$$

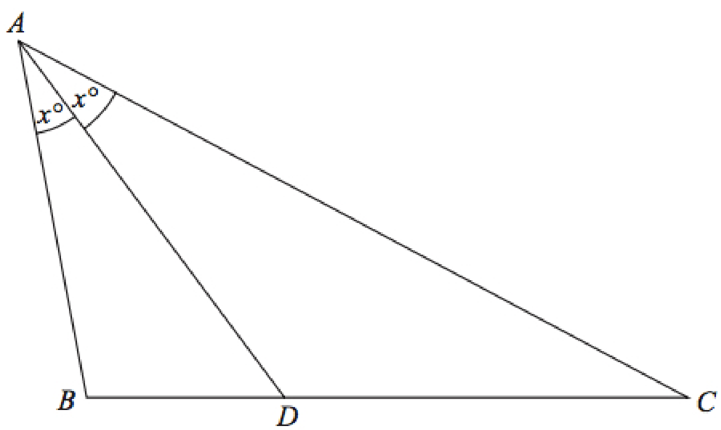
$$= 9(x + 4)(x - 4).$$

Now,

$$\frac{6x^3}{(9x^2-144)} \div \frac{2x^4}{3(x-4)} = \frac{6x^3}{9(x+4)(x-4)} \times \frac{3(x-4)}{2x^4}$$
$$= \underline{\underline{\frac{1}{x(x+4)}}};$$

hence, $\underline{\underline{r=4}}$.

23. ABC is a triangle. (4)



D is the point on BC such that

$$\text{angle } BAD = \text{angle } DAC = x^\circ.$$

Prove that

$$\frac{AB}{BD} = \frac{AC}{DC}.$$

Solution

We use the sine rule: in $\triangle ABD$,

$$\frac{\sin x^\circ}{BD} = \frac{\sin ADB}{AB} \Rightarrow \sin x^\circ = \frac{BD \sin ADB}{AB},$$

and, in $\triangle ACD$,

$$\frac{\sin x^\circ}{DC} = \frac{\sin ADC}{AC} \Rightarrow \sin x^\circ = \frac{DC \sin ADC}{AC}.$$

Put them together:

$$\frac{BD \sin ADB}{AB} = \frac{DC \sin ADC}{AC}.$$

But, both $\angle ADB$ and $\angle ADC$ supplementary angles and so $\sin ADB = \sin ADC$:

$$\begin{aligned}\frac{BD \sin ADB}{AB} = \frac{DC \sin ADC}{AC} &\Rightarrow \frac{BD}{AB} = \frac{DC}{AC} \\ &\Rightarrow \underline{\underline{\frac{AB}{BD} = \frac{AC}{DC}}},\end{aligned}$$

as required.