Dr Oliver Mathematics
Mathematics: Higher
2025 Paper 1: Non-Calculator
1 hour 15 minutes

The total number of marks available is 55.
You must write down all the stages in your working.

1. A curve has equation (4)

$$y = x^3 - 2x^2 + 5.$$

Find the equation of the tangent to this curve at the point where $x = 2$.

Solution

Well,

$$x = 2 \Rightarrow y = 8 - 8 + 5 = 5$$

so the point is $(2, 5)$.

Now

$$y = x^3 - 2x^2 + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x$$

and

$$\begin{aligned} x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 12 - 8 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 4. \end{aligned}$$

Finally, the equation of the tangent is

$$\begin{aligned} y - 5 = 4(x - 2) &\Rightarrow y - 5 = 4x - 8 \\ &\Rightarrow \underline{\underline{y = 4x - 3}}. \end{aligned}$$

2. Find the equation of the perpendicular bisector of the line joining $A(1, 4)$ and $B(9, 10)$. (4)

Solution

The midpoint — call it C — is

$$\left(\frac{1+9}{2}, \frac{4+10}{2}\right) = C(5, 7).$$

Now,

$$\begin{aligned} m_{AB} &= \frac{10-4}{9-1} \\ &= \tfrac{6}{8} \\ &= \tfrac{3}{4}, \end{aligned}$$

which means

$$m_{\text{normal}} = -\tfrac{4}{3}.$$

Finally, the equation of the perpendicular bisector is

$$\begin{aligned} y - 7 = -\tfrac{4}{3}(x-5) &\Rightarrow y - 7 = -\tfrac{4}{3}x + \tfrac{20}{3} \\ &\Rightarrow \underline{\underline{y = -\tfrac{4}{3}x + \tfrac{41}{3}}}. \end{aligned}$$

3. Find (4)

$$\int \left(\frac{12}{x^2} + x^{\frac{1}{2}} \right) \, \mathrm{d}x, \; x > 0.$$

Solution

Well,

$$\begin{aligned} \int \left(\frac{12}{x^2} + x^{\frac{1}{2}} \right) \, \mathrm{d}x &= \int \left(12x^{-2} + x^{\frac{1}{2}} \right) \, \mathrm{d}x \\ &= \underline{\underline{-12x^{-1} + \tfrac{2}{3}x^{\frac{3}{2}} + c}}. \end{aligned}$$

4. Evaluate (3)

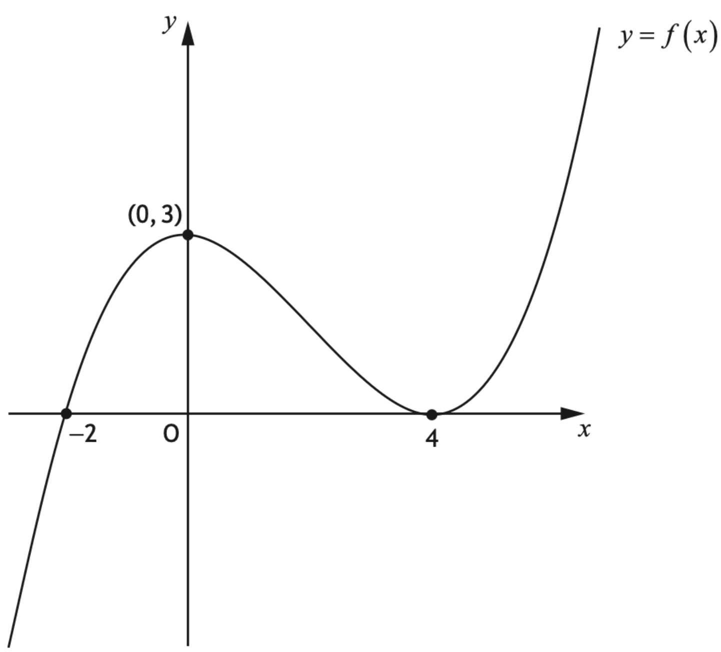
$$3 \log_3 2 + \log_3 \tfrac{1}{24}.$$

Solution

Now,

$$\begin{aligned}3\log_3 2 + \log_3 \tfrac{1}{24} &= \log_3 2^3 + \log_3 \tfrac{1}{24}\\ &= \log_3 8 + \log_3 \tfrac{1}{24}\\ &= \log_3(8 \times \tfrac{1}{24})\\ &= \log_3(\tfrac{1}{3})\\ &= \log_3(3^{-1})\\ &= -\log_3 3\\ &= \underline{\underline{-1}}.\end{aligned}$$

5. The diagram shows the graph of $y = \mathrm{f}(x)$, with stationary points at $(0, 3)$ and $(4, 0)$. (2)



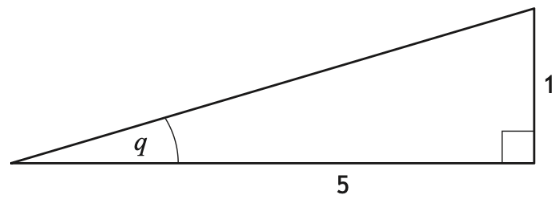
Sketch the graph of

$$y = \mathrm{f}(-x) + 3.$$

Solution

It goes through $(-4, 3)$, $(0, 6)$, and $(2, 3)$ and has a <u>down-up-down</u> shape.

6. The diagram shows a right-angled triangle with angle q.



(a) Determine the value of

(i) $\sin 2q$, (3)

Solution
Well,

$$\begin{aligned} \text{hyp}^2 = \text{opp}^2 + \text{adj}^2 &\Rightarrow \text{hyp}^2 = 1^2 + 5^2 \\ &\Rightarrow \text{hyp}^2 = 1 + 25 \\ &\Rightarrow \text{hyp}^2 = 26 \\ &\Rightarrow \text{hyp} = \sqrt{26}. \end{aligned}$$

Now,

$$\begin{aligned} \sin 2q &= 2 \sin q \cos q \\ &= 2 \times \tfrac{1}{\sqrt{26}} \times \tfrac{5}{\sqrt{26}} \\ &= \tfrac{10}{26} \\ &= \underline{\underline{\tfrac{5}{13}}}. \end{aligned}$$

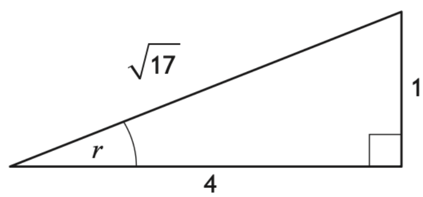
(ii) $\cos 2q$. (1)

Solution

Well,

$$\begin{aligned}\cos 2q &= 2\cos^2 q - 1\\ &= 2\cdot[\cos q]^2 - 1\\ &= 2\cdot\left[\tfrac{5}{\sqrt{26}}\right]^2 - 1\\ &= 2\cdot\left[\tfrac{25}{26}\right] - 1\\ &= \tfrac{50}{26} - 1\\ &= \tfrac{25}{13} - 1\\ &= \underline{\underline{\tfrac{12}{13}}}.\end{aligned}$$

A second right-angled triangle has angle r as shown.



(b) Find the value of $\sin(2q - r)$. (3)

Solution
Well,

$$\begin{aligned}\sin(2q-r) &= \sin(2q)\cos r - \cos(2q)\sin r\\ &= (\tfrac{5}{13})(\tfrac{4}{\sqrt{17}}) - (\tfrac{12}{13})(\tfrac{1}{\sqrt{17}})\\ &= \tfrac{20}{13\sqrt{17}} - \tfrac{12}{13\sqrt{17}}\\ &= \tfrac{8}{13\sqrt{17}}\\ &= \tfrac{8}{13\sqrt{17}} \times \tfrac{\sqrt{17}}{\sqrt{17}}\\ &= \tfrac{8\sqrt{17}}{13\times 17}\\ &= \underline{\underline{\tfrac{8\sqrt{17}}{221}}},\end{aligned}$$

using the trick of

$$\begin{aligned}13 \times 17 &= (15-2)(15+2)\\ &= 15^2 - 2^2\\ &= 225 - 4\\ &= 221.\end{aligned}$$

7. (a) Show that $(x+3)$ is a factor of (2)

$$5x^3 + 16x^2 - x - 12.$$

Solution
We use synthetic division:

$$\begin{array}{r|rrrr} -3 & 5 & 16 & -1 & -12\\ & \downarrow & -15 & -3 & 12\\ \hline & 5 & 1 & -4 & 0 \end{array}$$

As their is no remainder, $(x+3)$ is a factor of the cubic.

(b) Hence, or otherwise, solve (3)

$$5x^3 + 16x^2 - x - 12 = 0.$$

Solution
Now,

$$5x^3 + 16x^2 - x - 12 = 0 \Rightarrow (x+3)(5x^2 + x - 4) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & +1 \\ \text{multiply to:} & (+5)\times(-4) = -20\end{array}\right\} +5,\ -4$$

e.g.,

$$\begin{aligned}
&\Rightarrow (x+3)[5x^2+5x-4x-4] = 0 \\
&\Rightarrow (x+3)[5x(x+1)-4(x+1)] = 0 \\
&\Rightarrow (x+3)(5x-4)(x+1) = 0 \\
&\Rightarrow x+3=0,\ 5x-4=0, \text{ or } x+1=0 \\
&\Rightarrow \underline{\underline{x=-3,\ x=\tfrac{4}{5}, \text{ or } x=-1}}.
\end{aligned}$$

8. Given that (3)

$$\log_a 75 = 2 + \log_a 3,\ a > 0,$$

find the value of a.

Solution

Well,

$$\begin{aligned}
\log_a 75 = 2 + \log_a 3 &\Rightarrow \log_a 75 - \log_a 3 = 2 \\
&\Rightarrow \log_a\left(\frac{75}{3}\right) = 2 \\
&\Rightarrow \log_a 25 = 2 \\
&\Rightarrow a^2 = 25 \\
&\Rightarrow \underline{\underline{a=5}},
\end{aligned}$$

because $a > 0$.

9. Find the coordinates of the points of intersection of the line with equation (4)

$$y = x + 1$$

and the circle with equation

$$x^2 + y^2 - 2x + 6y - 15 = 0.$$

Solution

Well,

$\times$	x	$+1$
x	x^2	$+x$
$+1$	$+x$	$+1$

and so

$$\begin{aligned}
& x^2+y^2-2x+6y-15=0 \\
\Rightarrow\quad & x^2+(x+1)^2-2x+6(x+1)-15=0 \\
\Rightarrow\quad & x^2+(x^2+2x+1)-2x+6x+6-15=0 \\
\Rightarrow\quad & 2x^2+6x-8=0 \\
\Rightarrow\quad & 2(x^2+3x-4)=0
\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +3 \\ \text{multiply to:} & -4\end{array}\right\} +4, -1$$

$$\begin{aligned}
\Rightarrow\quad & 2(x+4)(x-1)=0 \\
\Rightarrow\quad & x+4=0 \text{ or } x-1=0 \\
\Rightarrow\quad & x=-4 \text{ or } x=1 \\
\Rightarrow\quad & y=-3 \text{ or } y=2;
\end{aligned}$$

hence, the coordinates are $\underline{\underline{(-4,-3)}}$ and $\underline{\underline{(1,2)}}$.

10. The vectors **u** and **v** are such that (5)

- $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,
- $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$, and
- the angle between **u** and **v** is 45°.

Find the value of k, where $k > 0$.

Solution

Well,

$$\begin{aligned}\mathbf{u}.\mathbf{v} &= (1)(1) + (1)(3) + (0)(k) \\ &= 4.\end{aligned}$$

Now,

$$\begin{aligned}|\mathbf{u}| &= \sqrt{1^2 + 1^2 + 0^2} \\ &= \sqrt{2}\end{aligned}$$

and

$$\begin{aligned}|\mathbf{v}| &= \sqrt{1^2 + 3^2 + k^2} \\ &= \sqrt{k^2 + 10}.\end{aligned}$$

Next,

$$\begin{aligned} & \mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos 45^\circ \\ \Rightarrow\quad & 4 = \sqrt{2} \times \sqrt{k^2 + 10} \times \tfrac{1}{\sqrt{2}} \\ \Rightarrow\quad & \sqrt{k^2 + 10} = 4 \\ \Rightarrow\quad & k^2 + 10 = 16 \\ \Rightarrow\quad & k^2 = 6 \\ \Rightarrow\quad & \underline{\underline{k = \sqrt{6}}}, \end{aligned}$$

as $k > 0$.

11. The equation (4)

$$9x^2 + 3kx + k = 0$$

has two real and distinct roots.

Determine the range of values for k.
Justify your answer.

Solution

Well,

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow (3k)^2 - 4(9)(k) > 0 \\ &\Rightarrow 9k^2 - 36k > 0 \\ &\Rightarrow 9k(k-4) > 0. \end{aligned}$$

We need a 'table of signs':

	$k<0$	$k=0$	$0<k<4$	$k=4$	$k>0$
k	$-$	0	$+$	$+$	$+$
$(k-4)$	$-$	$-$	$-$	0	$+$
$k(k-4)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{k < 0 \text{ or } k > 4}}.$$

12. Given that (4)

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6\cos x + 8\sin 2x$ and
- $y = 4$ when $x = \frac{1}{6}\pi$,

express y in terms of x.

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\cos x + 8\sin 2x \Rightarrow y = 6\sin x - 4\cos 2x + c,$$

for some constant c. Now,

$$\begin{aligned} x = \tfrac{1}{6}\pi, y = 4 &\Rightarrow 4 = 6\sin\tfrac{1}{6}\pi - 4\cos\tfrac{1}{3}\pi + c \\ &\Rightarrow 4 = 3 - 2 + c \\ &\Rightarrow c = 3; \end{aligned}$$

hence,

$$\underline{\underline{y = 6\sin x - 4\cos 2x + 3}}.$$

13. A function, f, is defined on the set of real numbers.

The derivative of f is

$$f'(x) = (x + 5)(2 - x).$$

(a) Find the x-coordinates of the stationary points on the curve with equation $y = f(x)$ and determine their nature. (3)

Solution

Well,

$\times$	x	$+5$
2	$2x$	$+10$
$-x$	$-x^2$	$-5x$

and

$$f'(x) = -x^2 - 3x + 10 \Rightarrow f''(x) = -2x - 3.$$

Now,

$$\begin{aligned} f'(x) = 0 &\Rightarrow (x + 5)(2 - x) = 0 \\ &\Rightarrow x + 5 = 0 \text{ or } 2 - x = 0 \\ &\Rightarrow x = -5 \text{ or } x = 2. \end{aligned}$$

Next,

$$x = -5 \Rightarrow f''(x) = 7 > 0$$

and

$$x = 2 \Rightarrow f''(x) = -7 < 0.$$

Hence, the stationary point at $\underline{\underline{x = -5}}$ is a minimum turning point and the stationary point at $\underline{\underline{x = 2}}$ is a maximum turning point.

It is known that

- f is a cubic function.
- $f(0) < 0$.
- The equation $f(x) = 0$ has exactly one solution. The solution lies between -10 and 10.

(b) Draw a sketch of a possible graph of $y = f(x)$. (3)

Solution

Now,

$$f'(x) = -x^2 - 3x + 10 \Rightarrow f(x) = -\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 10x + c,$$

for some constant c. So a graph looks like this: