Dr Oliver Mathematics
Advanced Level Paper 32: Mechanics
June 2022: Calculator
2 hours

The total number of marks available is 50.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.
(It goes with Paper 31: Statistics)

1. **In this question, position vectors are given relative to a fixed origin.**

At time t seconds, where $t > 0$, a particle P has velocity v ms^{-1} where

$$\mathbf{v} = 3t^2\mathbf{i} - 6\sqrt{t}\mathbf{j}.$$

(a) Find the speed of P at time $t = 2$ seconds. (2)

Solution

$$\begin{aligned} t = 2 \Rightarrow \mathbf{v}(2) &= 12\mathbf{i} - 6\sqrt{2}\mathbf{j} \\ \Rightarrow |\mathbf{v}(2)| &= \sqrt{12^2 + (-6\sqrt{2})^2} \\ \Rightarrow |\mathbf{v}(2)| &= 6\sqrt{6} \text{ or } 14.7 \text{ ms}^{-1} \text{ (3 sf)}. \end{aligned}$$

(b) Find an expression, in terms of t, $\mathbf{i}$, and $\mathbf{j}$, for the acceleration of P at time t seconds, where $t > 0$. (2)

Solution

$$\mathbf{v} = 3t^2\mathbf{i} - 6t^{\frac{1}{2}}\mathbf{j} \Rightarrow \mathbf{a} = (6t\mathbf{i} - 3u^{-\frac{1}{2}}\mathbf{j}) \text{ ms}^{-2}.$$

At time $t = 4$ seconds, the position vector of P is $(\mathbf{i} - 4\mathbf{j})$ m.

(c) Find the position vector of P at time $t = 1$ second. (4)

Solution

$$\mathbf{v} = 3t^2\mathbf{i} - 6t^{\frac{1}{2}}\mathbf{j} \Rightarrow \mathbf{s} = t^3\mathbf{i} - 4t^{\frac{3}{2}}\mathbf{j} + \mathbf{k},$$

for some vector $\mathbf{k}$. Now,

$$\begin{aligned} t = 4, P = \mathbf{i} - 4\mathbf{j} &\Rightarrow 64\mathbf{i} - 32\mathbf{j} + \mathbf{k} = \mathbf{i} - 4\mathbf{j} \\ &\Rightarrow \mathbf{k} = -63\mathbf{i} + 28\mathbf{j} \end{aligned}$$

and, hence,

$$\mathbf{s} = (t^3 - 63)\mathbf{i} + (-4t^{\frac{3}{2}} + 28)\mathbf{j}.$$

Finally,

$$t = 1 \Rightarrow \underline{\underline{\mathbf{s} = (-62\mathbf{i} + 24\mathbf{j})\text{ m}}}.$$

2. A rough plane is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$.

A small block B of mass 5 kg is held in equilibrium on the plane by a horizontal force of magnitude X newtons, as shown in Figure 1.

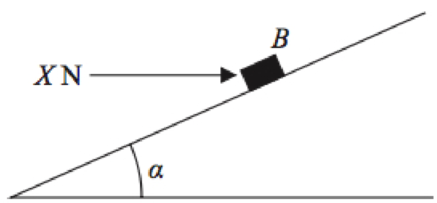

Figure 1: a rough plane

The force acts in a vertical plane which contains a line of greatest slope of the inclined plane.

The block B is modelled as a particle.

The magnitude of the normal reaction of the plane on B is 68.6 N.

Using the model,

(a) (i) find the magnitude of the frictional force acting on B, (3)

Solution
Let F N be the frictional force. Well,

$$\sin\alpha = \tfrac{3}{5} \text{ and } \cos\alpha = \tfrac{4}{5}$$

and

$$\begin{aligned} \text{Parallel:} \quad & X\cos\alpha + F - 5g\sin\alpha = 0 \\ \text{Perpendicular:} \quad & 68.8 = 5g\cos\alpha + X\sin\alpha. \end{aligned}$$

Now,

$$\begin{aligned} 68.6 = 5g\cos\alpha + X\sin\alpha &\Rightarrow 68.6 = 5g(\tfrac{4}{5}) + \tfrac{3}{5}X \\ &\Rightarrow 68.6 = 4g + \tfrac{3}{5}X \\ &\Rightarrow \tfrac{3}{5}X = 29.4 \\ &\Rightarrow X = 49 \end{aligned}$$

and

$$\begin{aligned} X\cos\alpha + F - 5g\sin\alpha = 0 &\Rightarrow (49)(\tfrac{4}{5}) - F - 5g(\tfrac{3}{5}) = 0 \\ &\Rightarrow 39.2 + F - 3g = 0 \\ &\Rightarrow F = -9.8, \end{aligned}$$

and we drew frictional in the wrong direction!

Hence, the magnitude of the frictional force acting on B is 9.8 N.

(ii) state the direction of the frictional force acting on B. (1)

Solution
Down the slope.

The horizontal force of magnitude X newtons is now removed and B moves down the plane. Given that the coefficient of friction between B and the plane is 0.5,

(b) find the acceleration of B down the plane. (6)

Solution

$$\begin{aligned} \text{Parallel:} \quad & 5a = 5g\sin\alpha - F \\ \text{Perpendicular:} \quad & R = 5g\cos\alpha \\ F = \mu R: \quad & F = 0.5R. \end{aligned}$$

Now,

$$\begin{aligned} 5a = 5g\sin\alpha - F &\Rightarrow 5a = 5g(\tfrac{3}{5}) - 0.5R \\ &\Rightarrow 5a = 3g - 0.5(5g\cos\alpha) \\ &\Rightarrow 5a = 3g - 2.5g(\tfrac{4}{5}) \\ &\Rightarrow 5a = 3g - 2g \\ &\Rightarrow 5a = g \\ &\Rightarrow \underline{\underline{a = \tfrac{1}{5}g}}. \end{aligned}$$

3. **In this question, i and j are horizontal unit vectors.**

A particle P of mass 4 kg is at rest at the point A on a smooth horizontal plane. At time $t = 0$, two forces,

$$F_1 = (4\mathbf{i} - \mathbf{j}) \text{ N and } F_2 = (\lambda\mathbf{i} + \mu\mathbf{j}) \text{ N},$$

where λ and μ are constants, are applied to P.

Given that P moves in the direction of the vector $(3\mathbf{i} + \mathbf{j})$,

(a) show that (4)

$$\lambda - 3\mu + 7 = 0.$$

Solution

After the two forces are applied, P moves in the direction of the vector

$$(4 + \lambda)\mathbf{i} + (-1 + \mu)\mathbf{j}$$

and this vector is parallel to $(3\mathbf{i} + \mathbf{j})$:

$$\begin{aligned} 4 + \lambda &= 3k \quad (1) \\ -1 + \mu &= k \quad (2), \end{aligned}$$

for some constant k. Do $3 \times (2)$:

$$-3 + 3\mu = 3k \quad (3)$$

and $(1) - (3)$:

$$\begin{aligned} (4 + \lambda) - (-3 + 3\mu) = 0 &\Rightarrow 4 + \lambda + 3 - 3\mu = 0 \\ &\Rightarrow \underline{\underline{\lambda - 3\mu + 7 = 0}}, \end{aligned}$$

as required.

At time $t = 4$ seconds, P passes through the point B.

Given that $\lambda = 2$,

(b) find the length of AB. (5)

Solution

$$\begin{aligned}\lambda = 2 &\Rightarrow 2 - 3\mu + 7 = 0\\ &\Rightarrow 3\mu = 9\\ &\Rightarrow \mu = 3\end{aligned}$$

so the two forces are acting with a force of

$$(6\mathbf{i} + 2\mathbf{j})\text{ N}.$$

Let $\mathbf{a}$ ms^{-2} be the acceleration. Now,

$$(6\mathbf{i} + 2\mathbf{j}) = 4\mathbf{a} \Rightarrow \mathbf{a} = (1.5\mathbf{i} + 0.5\mathbf{j}).$$

Next, $\mathbf{s} =?$, $\mathbf{u} = \mathbf{0}$, $\mathbf{v} =?$, $\mathbf{a} = 1.5\mathbf{i} + 0.5\mathbf{j}$, $t = 4$:

$$\begin{aligned}\mathbf{s} &= \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2\\ &= \mathbf{0} + \tfrac{1}{2}(1.5\mathbf{i} + 0.5\mathbf{j})(4^2)\\ &= 12\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

Finally,

$$\begin{aligned}AB &= \sqrt{12^2 + 4^2}\\ &= \underline{\underline{4\sqrt{10}\text{ or }12.6\text{ m (3 sf)}}}\end{aligned}$$

4. A uniform rod AB has mass M and length $2a$.

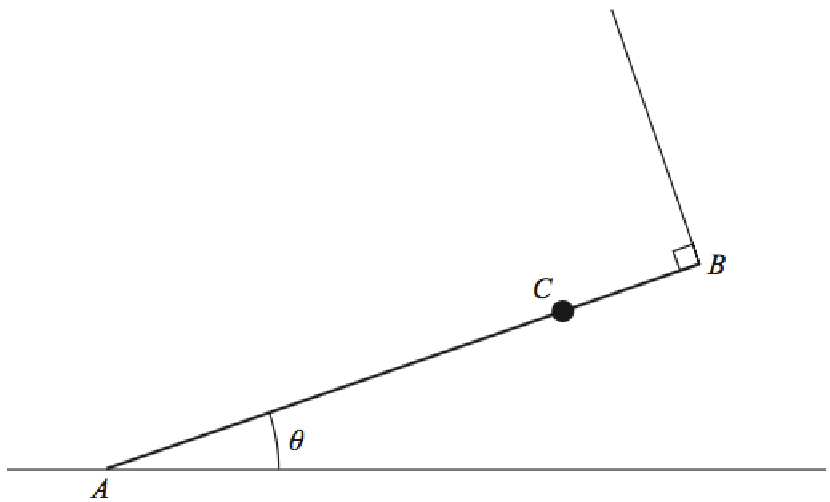

Figure 2: a uniform rod AB has mass M and length $2a$

A particle of mass $2M$ is attached to the rod at the point C, where $AC = 1.5a$.

The rod rests with its end A on rough horizontal ground.

The rod is held in equilibrium at an angle θ to the ground by a light string that is attached to the end B of the rod.

The string is perpendicular to the rod, as shown in Figure 2.

(a) Explain why the frictional force acting on the rod at A acts horizontally to the right on the diagram. (1)

> **Solution**
> E.g., the horizontal component of T acts to the *left* and since the only other horizontal force is friction, it must act to the right.

The tension in the string is T.

(b) Show that (3)

$$T = 2Mg\cos\theta.$$

> **Solution**
>
> Moments about A: $(T)(2a) = (Mg)(a\cos\theta) + (2Mg)(1.5a\cos\theta).$

Now,

$$\begin{aligned}
&(T)(2a) = (Mg)(a\cos\theta) + (2Mg)(1.5a\cos\theta) \\
\Rightarrow\quad & 2aT = aMg\cos\theta + 3aMg\cos\theta \\
\Rightarrow\quad & 2aT = 4aMg\cos\theta \\
\Rightarrow\quad & \underline{\underline{T = 2Mg\cos\theta}},
\end{aligned}$$

as required.

Given that $\cos\theta = \frac{3}{5}$,

(c) show that the magnitude of the vertical force exerted by the ground on the rod at A is (3)

$$\frac{57Mg}{25}.$$

Solution

$$R(\updownarrow):\quad R + T\cos\theta = Mg + 2Mg$$

Now,

$$\begin{aligned}
R + T\cos\theta = Mg + 2Mg &\Rightarrow R + (2Mg \cdot \tfrac{3}{5})(\tfrac{3}{5}) = 3Mg \\
&\Rightarrow R + \tfrac{18}{25}Mg = 3Mg \\
&\Rightarrow \underline{\underline{R = \tfrac{57}{25}Mg}},
\end{aligned}$$

as required.

The coefficient of friction between the rod and the ground is μ.

Given that the rod is in limiting equilibrium,

(d) show that (4)

$$\mu = \tfrac{8}{19}.$$

Solution

Well,

$$\sin\theta = \tfrac{4}{5}$$

and

$$R(\leftrightarrow): \quad F = T\sin\theta$$
$$\text{Limiting equilibrium}: \quad F = \mu R$$

Now,

$$\begin{aligned} F = \mu R &\Rightarrow \tfrac{4}{5}T = \mu\left(\tfrac{57}{25}Mg\right) \\ &\Rightarrow \tfrac{4}{5}T = \mu\left(\tfrac{57}{25}Mg\right) \\ &\Rightarrow \tfrac{4}{5}\left(2Mg\cdot\tfrac{3}{5}\right) = \mu\left(\tfrac{57}{25}Mg\right) \\ &\Rightarrow \tfrac{12}{25} = \mu\left(\tfrac{57}{25}\right) \\ &\Rightarrow \underline{\underline{\mu = \tfrac{4}{19}}}, \end{aligned}$$

as required.

5. A golf ball is at rest at the point A on horizontal ground.

The ball is hit and initially moves at an angle α to the ground.

The ball first hits the ground at the point B, where $AB = 120$ m, as shown in Figure 3.

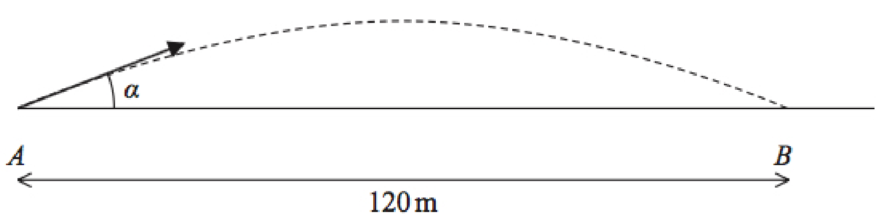

Figure 3: a golf ball

The motion of the ball is modelled as that of a particle, moving freely under gravity, whose initial speed is U ms^{-1}.

Using this model,

(a) show that (6)

$$U^2 \sin\alpha\cos\alpha = 588.$$

Solution

Horizontally:

$\rightarrow$: $s = 120$, $u = U\cos\alpha$, $v = U\cos\alpha$, $a = 0$, and $t = ?$.

Well,

$$s = ut + \tfrac{1}{2}at^2 \Rightarrow 120 = tU\cos\alpha$$
$$\Rightarrow t = \frac{120}{U\cos\alpha}.$$

Now, vertically:

$\uparrow$: $s = 0$, $u = U\sin\alpha$, $v = -U\sin\alpha$, $a = -9.8$, and $t = \dfrac{120}{U\cos\alpha}$.

Now,

$$v = u + at \Rightarrow -U\sin\alpha = U\sin\alpha + (-9.8)\left(\frac{120}{U\cos\alpha}\right)$$
$$\Rightarrow 2U\sin\alpha = \frac{1\,176}{U\cos\alpha}$$
$$\Rightarrow 2U^2\sin\alpha\cos\alpha = 1\,176$$
$$\Rightarrow \underline{\underline{U^2\sin\alpha\cos\alpha = 588}},$$

as required.

The ball reaches a maximum height of 10 m above the ground.

(b) Show that (4)

$$U^2 = 1\,960.$$

Solution

$\uparrow$: $s = 10$, $u = U\sin\alpha$, $v = 0$, $a = -9.8$, and $t = ?$:

$$v^2 = u^2 + 2as \Rightarrow 0 = (U\sin\alpha)^2 + 2(-9.8)(10)$$
$$\Rightarrow U^2\sin^2\alpha = 196.$$

Now, we know two expressions for U^2:

$$U^2\sin^2\alpha = 196 \quad (1)$$
$$U^2\sin\alpha\cos\alpha = 588 \quad (2)$$

and so (1) ÷ (2):

$$\frac{U^2 \sin^2 \alpha}{U^2 \sin \alpha \cos \alpha} = \frac{196}{588} \Rightarrow \tan \alpha = \tfrac{1}{3}$$
$$\Rightarrow \tan \alpha = \tfrac{1}{3}$$
$$\Rightarrow \sin \alpha = \tfrac{1}{\sqrt{10}}$$
$$\Rightarrow \sin^2 \alpha = \tfrac{1}{10},$$

and, finally,

$$U^2 \sin^2 \alpha = 196 \Rightarrow \tfrac{1}{10} U^2 = 196$$
$$\Rightarrow \underline{\underline{U^2 = 1\,960}},$$

In a refinement to the model, the effect of air resistance is included.

The motion of the ball, from A to B, is now modelled as that of a particle whose initial speed is V ms^{-1}.

This refined model is used to calculate a value for V.

(c) State which is greater, U or V, giving a reason for your answer. (1)

Solution
$\underline{\underline{V}}$ is greater, since air resistance has to be overcome.

(d) State one further refinement to the model that would make the model more realistic. (1)

Solution
E.g., spin of the ball, size of the ball, the effect of the wind, it would have variable acceleration.