Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2010 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Solve the inequality (3)

$$3 - x < 4(x - 1).$$

Solution

$$\begin{aligned} 3 - x < 4(x-1) &\Rightarrow 3 - x < 4x - 4 \\ &\Rightarrow 7 < 5x \\ &\Rightarrow \underline{\underline{x > 1\tfrac{2}{5}}}. \end{aligned}$$

2. Expand (3)

$$(1 - x)^{12}$$

in ascending powers of x up to the term in x^3, and simplify your answer.

Solution

$$\begin{aligned} (1-x)^{12} &= [1 + (-x)]^{12} \\ &= 1 + \binom{12}{1}(-x) + \binom{12}{2}(-x)^2 + \binom{12}{3}(-x)^3 + \dots \\ &= \underline{\underline{1 - 12x + 66x^2 - 220x^3 + \dots}}. \end{aligned}$$

3. The function $f(x)$ is defined by

$$f(x) = x^3 - 5x^2 + 2x + 8.$$

(a) Find the remainder when $f(x)$ is divided by $(x+1)$. (2)

Solution

We use synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -5 & 2 & 8 \\ & \downarrow & -1 & 6 & -8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

Hence, the remainder is $\underline{\underline{0}}$.

(b) Solve the equation $f(x) = 0$. (3)

Solution

$$x^3 - 5x^2 + 2x + 8 = 0 \Rightarrow (x+1)(x^2 - 6x + 8) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -6 \\ \text{multiply to:} & +8 \end{array}\right\} -4, -2$$

$$\begin{aligned} &\Rightarrow (x+1)(x-4)(x-2) = 0 \\ &\Rightarrow \underline{\underline{x = -1, x = 2, \text{ or } x = 4}}. \end{aligned}$$

4. In a game 4 fair dice are thrown.

Calculate the probability that

(a) no six is thrown, (2)

Solution

$$\begin{aligned} \text{P(no six is thrown)} &= \left(\tfrac{5}{6}\right)^4 \\ &= \underline{\underline{\tfrac{625}{1\,296} \text{ or } 0.482\,3 \text{ (4 dp)}}}. \end{aligned}$$

(b) at least 2 sixes are thrown. (4)

Solution

$$\begin{aligned}\text{P(at least 2 sixes are thrown)} &= \text{P(2 sixes)} + \text{P(3 sixes)} + \text{P(4 sixes)} \\ &= \binom{4}{2}(\tfrac{5}{6})^2(\tfrac{1}{6})^2 + \binom{4}{3}(\tfrac{5}{6})(\tfrac{1}{6})^3 + (\tfrac{1}{6})^4 \\ &= \tfrac{25}{216} + \tfrac{5}{324} + \tfrac{1}{1\,296} \\ &= \underline{\underline{\tfrac{19}{144} \text{ or } 0.131\,9 \text{ (4 dp)}}}.\end{aligned}$$

5. The curve

$$y = x^3 - 3x^2 - 9x + 7$$

has two turning points, one of which is where $x = 3$.

(a) Find the coordinates of the other turning point and determine whether it is a maximum or minimum point. (5)

Solution

$$\begin{aligned}y = x^3 - 3x^2 - 9x + 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 6x - 9 \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6x - 6\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 6x - 9 = 0 \\ &\Rightarrow 3(x^2 - 2x - 3) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -2 \\ \text{multiply to:} & -3\end{array}\right\} -3, +$$

$$\begin{aligned}&\Rightarrow 3(x - 3)(x + 1) = 0 \\ &\Rightarrow x = 3 \text{ or } x = -1 \\ &\Rightarrow y = -20 \text{ or } y = 12.\end{aligned}$$

Now,

$$x = -1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -12 < 0$$

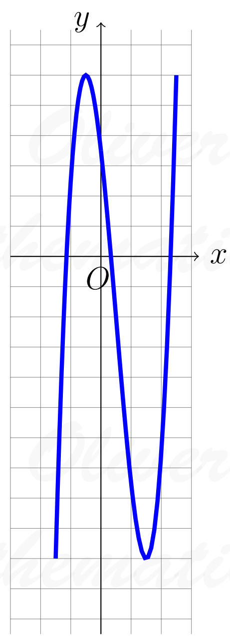
and so $\underline{\underline{(-1, 12)}}$ is a $\underline{\underline{\text{minimum turning point}}}$ and

$$x = 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12 > 0$$

and so $\underline{\underline{(3, -20)}}$ is a $\underline{\underline{\text{maximum turning point}}}$.

(b) Sketch the curve. (1)

Solution



6. An aeroplane touches down at a point A on a runway, travelling at 90 ms^{-1}. It then decelerates uniformly until it reaches a speed of 6 ms^{-1} at a point B on the runway, 2016 m from A.

(a) Find the deceleration. (3)

Solution

$s = 2\,016$, $u = 90$, $v = 6$, $a = ?$, and $t = ?$: use $v^2 = u^2 + 2as$:

$$\begin{aligned} 6^2 = 90^2 + 2 \times a \times 2\,016 &\Rightarrow 36 = 8\,100 + 4\,032a \\ &\Rightarrow 4\,032a = -8\,064 \\ &\Rightarrow a = -2; \end{aligned}$$

hence, the deceleration is $\underline{\underline{2 \text{ ms}^{-2}}}$.

(b) Find the time taken to travel from A to B. (2)

Solution

Use $v = u + at$:

$$\begin{aligned} 6 = 90 + (-2)t &\Rightarrow -42t = -84 \\ &\Rightarrow \underline{\underline{t = 42 \text{ s}}}. \end{aligned}$$

7. It is required to solve the equation

$$\sin\theta\cos\theta = \tfrac{1}{4}.$$

(a) Show that (1)

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \equiv \frac{1}{\sin\theta\cos\theta}.$$

Solution

$$\begin{aligned} \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &\equiv \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \\ &\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ &\equiv \underline{\underline{\frac{1}{\sin\theta\cos\theta}}}, \end{aligned}$$

as required.

(b) Hence show that the equation (2)

$$\sin\theta\cos\theta = \tfrac{1}{4}$$

is equivalent to

$$\tan\theta + \frac{1}{\tan\theta} = 4.$$

Solution

$$\begin{aligned}\tan\theta + \frac{1}{\tan\theta} &= \frac{\sin\theta}{\cos\theta} + \frac{1}{\frac{\sin\theta}{\cos\theta}} \\ &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ &= \frac{1}{\sin\theta\cos\theta} \\ &= \frac{1}{\frac{1}{4}} \\ &= \underline{\underline{4}},\end{aligned}$$

as required.

(c) By expressing this equation as a quadratic equation in t, where $t = \tan\theta$, find the two values of θ, in the range $0^\circ \leqslant \theta \leqslant 180^\circ$, that satisfy the equation. (4)

Solution

$$\begin{aligned}\sin\theta\cos\theta = \tfrac{1}{4} &\Rightarrow t + \frac{1}{t} = 4 \\ &\Rightarrow t^2 + 1 = 4t \\ &\Rightarrow t^2 - 4t = -1 \\ &\Rightarrow t^2 - 4t + 4 = -1 + 4 \\ &\Rightarrow (t-2)^2 = 3 \\ &\Rightarrow t - 2 = \pm\sqrt{3} \\ &\Rightarrow t = 2 \pm \sqrt{3}.\end{aligned}$$

Now,

$$\tan\theta = 2 - \sqrt{3} \Rightarrow \underline{\underline{\theta = 15^\circ}}$$

and

$$\tan\theta = 2 + \sqrt{3} \Rightarrow \underline{\underline{\theta = 75^\circ}}.$$

8. A train moves between two stations, taking 5 minutes for the journey. (5)

The velocity of the train may be modelled by the equation

$$v = 60(t^4 - 10t^3 + 25t^2),$$

where v is measured in metres per minute and t is measured in minutes.

Calculate the distance between the two stations.

Solution

$$\begin{aligned}
\text{Distance} &= \int_0^5 60(t^4 - 10t^3 + 25t^2)\,\mathrm{d}t \\
&= \int_0^5 (60t^4 - 600t^3 + 1\,500t^2)\,\mathrm{d}t \\
&= \left[12t^5 - 150t^4 + 500t^3\right]_{t=0}^{5} \\
&= (37\,500 - 93\,750 + 62\,500) - (0 - 0 + 0) \\
&= \underline{\underline{6\,250\text{ m}}}.
\end{aligned}$$

9. The diameter of a circle is PQ, where P and Q are the points $(1, 3)$ and $(15, 1)$ respectively.

   (a) Find the centre of the circle. (2)

   Solution

   $$\left(\frac{1+13}{2}, \frac{3+1}{2}\right) = \underline{\underline{(8, 2)}}.$$

   (b) Show that the radius of the circle is $5\sqrt{2}$. (2)

   Solution

   $$\begin{aligned}
   \text{Radius} &= \sqrt{(8-1)^2 + (2-3)^2} \\
   &= \sqrt{49 + 1} \\
   &= \sqrt{50} \\
   &= \sqrt{25 \times 2} \\
   &= \sqrt{25} \times \sqrt{2} \\
   &= \underline{\underline{5\sqrt{2}}},
   \end{aligned}$$

   as required.

   (c) Hence find the equation of the circle in the form (2)

   $$x^2 + y^2 + ax + by + c = 0.$$

Solution

$$\begin{aligned}(x-8)^2+(y-2)^2=(5\sqrt{2})^2 &\Rightarrow (x^2-16x+64)+(y^2-4y+4)=50\\ &\Rightarrow \underline{\underline{x^2+y^2-16x-4y-18=0}};\end{aligned}$$

hence, $\underline{\underline{a=-16}}$, $\underline{\underline{b=-14}}$, and $\underline{\underline{c=-18}}$.

10. John and Paul are carrying out an experiment.

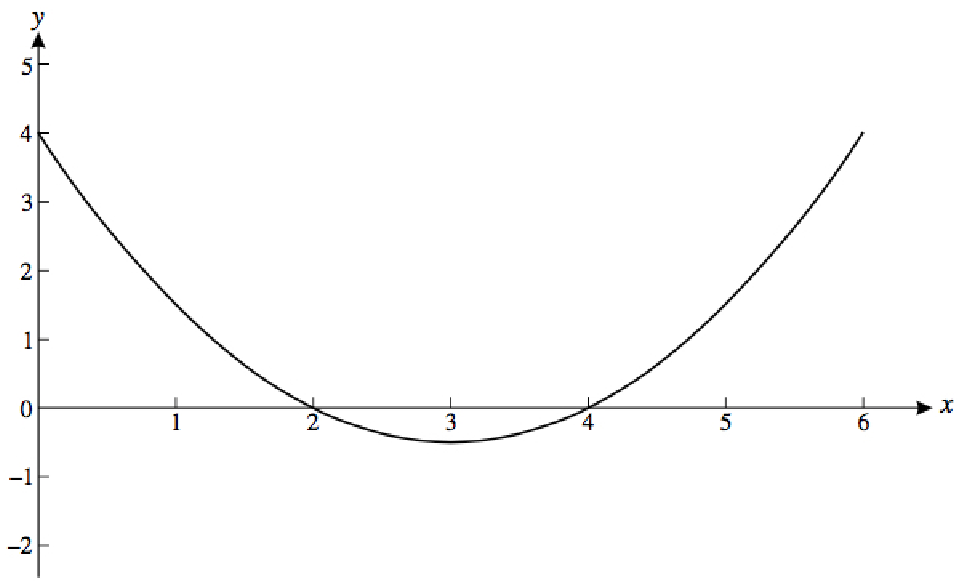
The table shows their results for x and y.

x	0	2	3	4
y	4	0	0.25	0

Paul proposes that the relationship should be modelled by

$$y=k(x-2)(x-4),$$

and this is indicated in the figure below.



(a) Find the value of k for which the points $(0,4)$, $(2,0)$, and $(4,0)$ satisfy this equation. (2)

Solution
As it goes through $(0, 4)$,

$$4 = k(0-2)(0-4) \Rightarrow 8k = 4 \Rightarrow \underline{\underline{k = \tfrac{1}{2}}}.$$

John proposes a different model, using

$$y = c(x-2)^2(x-4).$$

(b) Find the value of c for which the points $(0, 4)$, $(2, 0)$, and $(4, 0)$ satisfy this equation. (2)

Solution
As it goes through $(0, 4)$,

$$4 = c(0-2)^2(0-4) \Rightarrow -16c = 4 \Rightarrow \underline{\underline{c = -\tfrac{1}{4}}}.$$

(c) Which is the better model for John and Paul's results? Give a reason for your answer. (2)

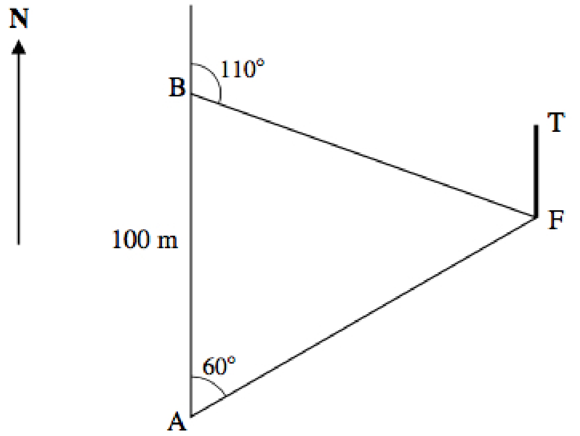
Solution
John: when

$$x = 3 \Rightarrow y = -\tfrac{1}{4}(3-2)^2(3-4) = 0.25$$

whereas Paul is negative.

Section B

11. Michael is at a point A and the base of a church tower is at a point F, as shown in the figure below.



He measures the bearing of the tower to be 060°.
Michael walks 100 metres due North to the point B from where he measures the bearing of F to be 110°.
The triangle ABF is in the horizontal plane.

(a) Show that $AF = 122.7$ m, correct to 4 significant figures, and find BF. (5)

Solution
Well,

$$\angle ABF = 180 - 110 = 70^\circ$$

and

$$\angle AFB = 180 - (60 + 70) = 50^\circ.$$

Now, ww apply the sine rule:

$$\begin{aligned}\frac{AF}{\sin ABF} = \frac{AB}{\sin AFB} &\Rightarrow \frac{AF}{\sin 70^\circ} = \frac{100}{\sin 50^\circ}\\ &\Rightarrow AF = \frac{100 \sin 70^\circ}{\sin 50^\circ}\\ &\Rightarrow AF = 122.668\,159\,7 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{AF = 122.7 \text{ m (4 sf)}}},\end{aligned}$$

as required. Now, we apply the cosine rule:

$$\begin{aligned}BE &= \sqrt{AB^{+}AF^2 - 2 \cdot AB \cdot AF \cdot \cos BAF}\\ &= \sqrt{100^{+}122.668\ldots^2 - 2 \cdot 100 \cdot 122.668\ldots \cdot \cos 60^\circ}\\ &= 113.051\,587\,5 \text{ (FCD)}\\ &= \underline{\underline{113.1 \text{ m (4 sf)}}},\end{aligned}$$

Michael finds that the angle of elevation of the top of the tower, T, from A is $10°$.

(b) Find the height of the tower. (2)

Solution

$$\frac{\text{opp}}{\text{adj}} = \tan \Rightarrow \frac{\text{height}}{122.668\ldots} = \tan 10^\circ$$
$$\Rightarrow \text{height} = 122.668\ldots \tan 10^\circ$$
$$\Rightarrow \text{height} = 21.629\,706\,23 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\text{height} = 21.6 \text{ m (3 sf)}}}.$$

C is the point on AB that is nearest to F.

(c) Find CF and the angle of elevation from C to the top of the tower, correct to 1 decimal place. (5)

Solution

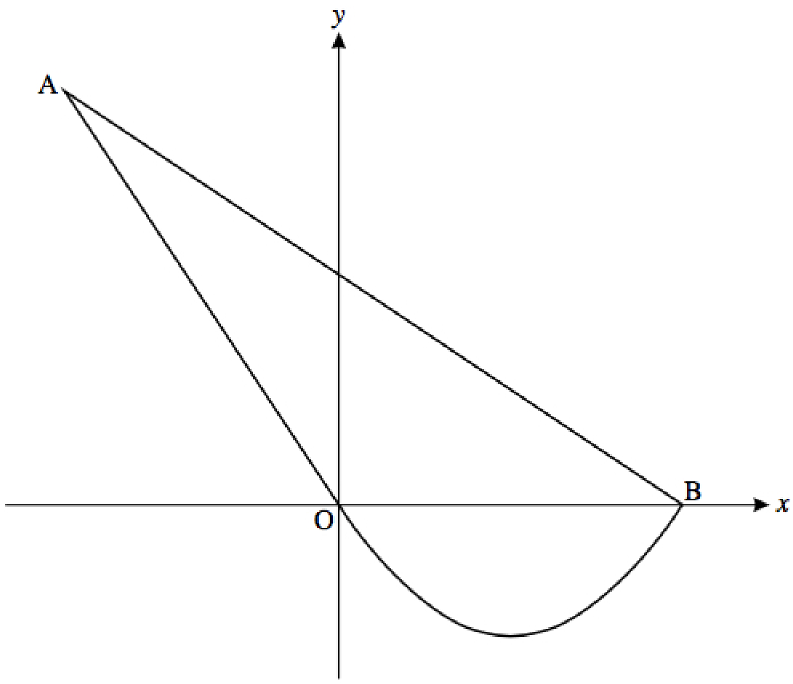
So, $\triangle ACF$ is a right-angled triangle with the $\angle ACF = 90°$. Now,

$$\frac{\text{opp}}{\text{hyp}} = \sin \Rightarrow \frac{CF}{122.668\ldots} = \sin 60^\circ$$
$$\Rightarrow CF = 122.668\ldots \sin 60^\circ$$
$$\Rightarrow CF = 106.233\,742\,5 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{CF = 106.2 \text{ m (4 sf)}}}.$$

Finally,

$$\frac{\text{opp}}{\text{adj}} = \tan \Rightarrow \tan(\text{angle of elevation}) = \frac{21.629\ldots}{106.233\ldots}$$
$$\Rightarrow \text{angle of elevation} = 11.508\,393\,37 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\text{angle of elevation} = 11.5^\circ \text{ (3 sf)}}}.$$

12. The figure shows the shape AOB that is to be made from card.



B is the point $(5, 0)$ and OB is part of the curve with equation

$$y = 0.3x^2 - 1.5x.$$

The line AB is the normal to the curve at B.

(a) Find the equation of the line AB. (4)

Solution

$$y = 0.3x^2 - 1.5x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0.6x - 1.5.$$

Now,

$$x = 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2} \Rightarrow m_{\text{normal}} = -\tfrac{2}{3}.$$

Finally,

$$y - 0 = -\tfrac{2}{3}(x - 5) \Rightarrow \underline{\underline{y = -\tfrac{2}{3}x + \tfrac{10}{3}}}.$$

The equation of the line AO is

$$2y + 3x = 0.$$

(b) Find the coordinates of the point A. (3)

Solution

$$2y + 3x = 0 \Rightarrow 2y = -3x \Rightarrow y = -\tfrac{3}{2}x$$

and we solve:

$$\begin{aligned} -\tfrac{2}{3}x + \tfrac{10}{3} = -\tfrac{3}{2}x &\Rightarrow -\tfrac{5}{6}x = \tfrac{10}{3} \\ &\Rightarrow x = -4 \\ &\Rightarrow y = 6; \end{aligned}$$

hence, $\underline{\underline{A(-4, 6)}}$.

(c) Find the area of the shape AOB. (5)

Solution

$$\text{Area} = \text{area of } \triangle AOB - \text{area under the curve.}$$

(Why do we have subtract? Hint: it is below the x-axis.) Now

$$\begin{aligned} \text{area of } \triangle AOB &= \tfrac{1}{2} \times 5 \times 6 \\ &= 15. \end{aligned}$$

Next,

$$\begin{aligned} \text{area under the curve} &= \int_0^5 (0.3x^2 - 1.5x)\,\mathrm{d}x \\ &= \left[0.1x^3 - 0.75^2\right]_{x=0}^5 \\ &= (12.5 - 18.75) - (0 - 0) \\ &= -6.25. \end{aligned}$$

Finally,

$$\text{area} = 15 - (-6.25) = \underline{\underline{21.25}}.$$

13. Ali and Beth make components in a factory. Ali works faster than Beth and makes 3 more components per hour. As a result he takes 2 hours less time than Beth to make 72 components.

Let t hours be the time that Ali takes to make 72 components.

(a) Write expressions for the numbers of components made per hour by Ali and by Beth. (3)

Solution

$$\text{Ali}: \underline{\underline{\frac{72}{t}}} \text{ and Beth}: \underline{\underline{\frac{72}{t+2}}}.$$

(b) Hence derive the equation (5)

$$3t(t+2) = 144.$$

Solution

'Ali works faster than Beth and makes 3 more components per hour' so

$$\begin{aligned}\frac{72}{t} - \frac{72}{t+2} = 3 &\Rightarrow 72(t+2) - 72t = 3t(t+2)\\ &\Rightarrow \underline{\underline{3t(t+2) = 144}},\end{aligned}$$

as required.

(c) Solve this equation to find the times that Ali and Beth take to make 72 components. (4)

Solution

$$\begin{aligned}3t(t+2) = 144 &\Rightarrow 3t^2 + 6t - 144 = 0\\ &\Rightarrow 3(t^2 + 2t - 48) = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +2\\ \text{multiply to:} & -48\end{array}\right\} -6, +8$$

$$\begin{aligned}&\Rightarrow 3(t-6)(t+8) = 0\\ &\Rightarrow t = -8 \text{ or } t = 6;\end{aligned}$$

clearly, $t > 0$: Ali takes $\underline{\underline{\text{6 hours}}}$ and Beth takes $\underline{\underline{\text{8 hours}}}$.

14. A firm has to transport 1500 packages to a site It has a number of large vans which will transport 200 packages each and a number of small vans which will transport 100 packages each.

Let x be the number of large vans and let y be the number of small vans used.

(a) Write down an inequality based on the number of packages transported. (2)

Solution

$\underline{\underline{200x + 100y \geqslant 1\,500}}$.

The firm needs to use at least as many small vans as large vans.

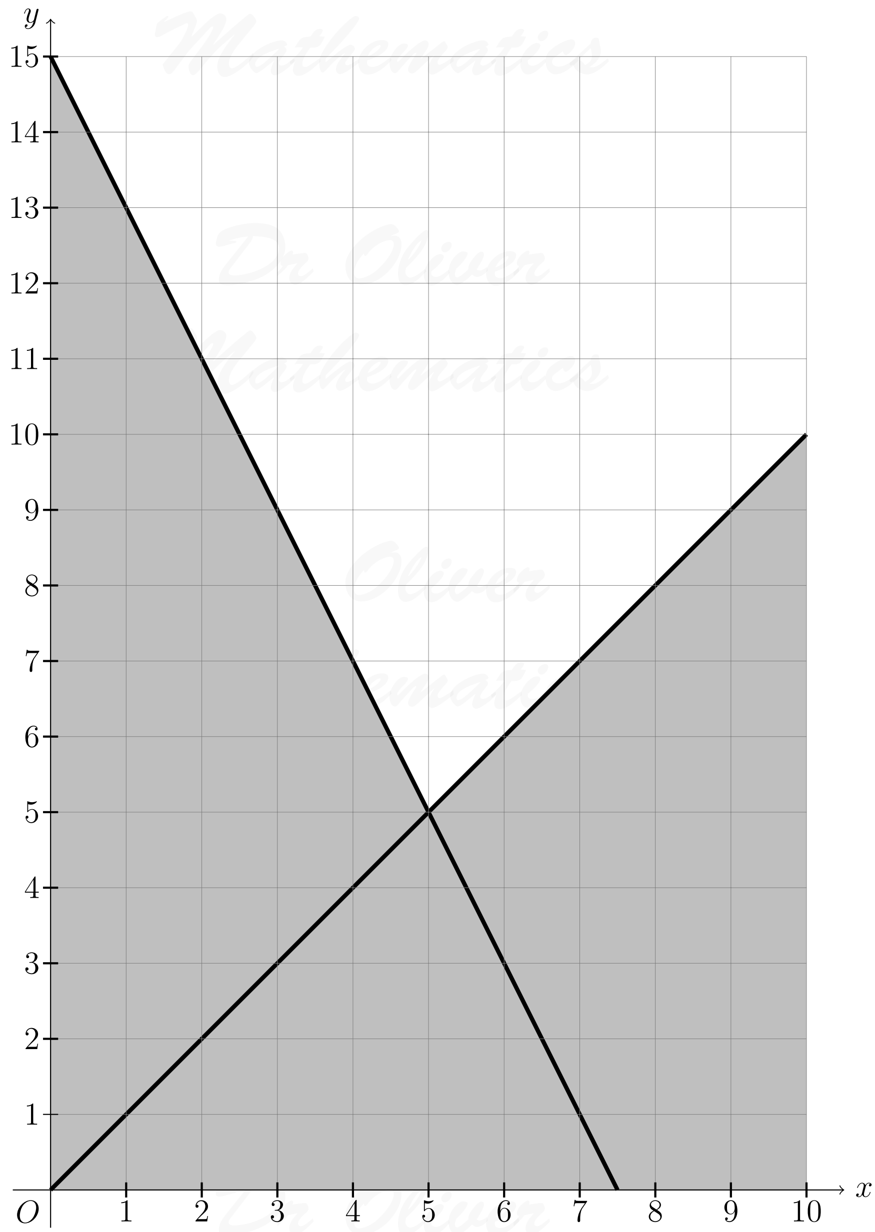
(b) Write a second inequality. (1)

Solution

$\underline{\underline{y \geqslant x}}$.

(c) Plot these two inequalities on a graph, using 1 cm to represent one van on each axis. Indicate the region for which these inequalities hold. Shade the area that is **not** required. (3)

Solution



A large van costs £80 to complete the trip and a small van costs £60 to complete the trip.

(d) Write down the objective function and hence find from your graph the number of each type of van that will minimise the cost, and work out that cost. (4)

Solution

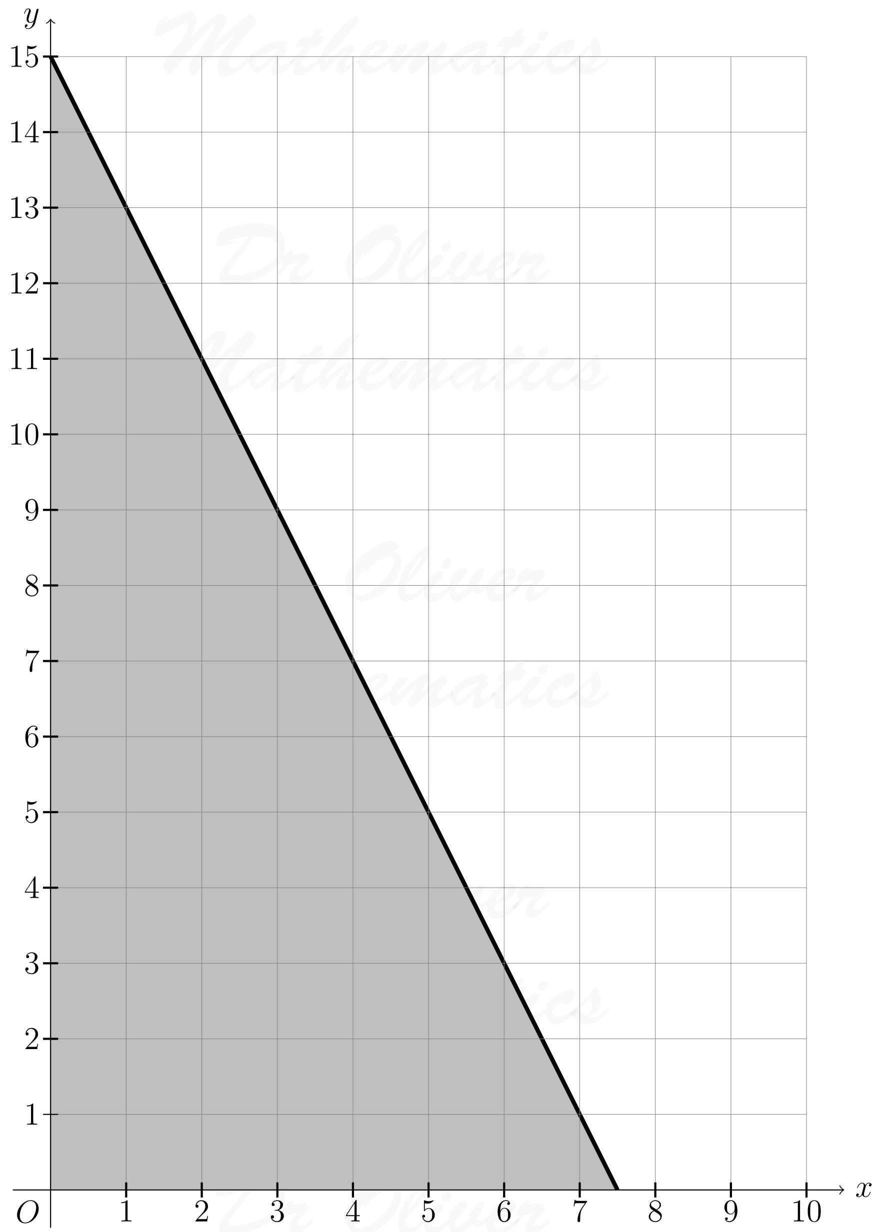
$C = 80x + 60y$.

From the graph, $(5, 5)$ is clearly the minimum cost:

$$C = 80(5) + 60(5) = \underline{\underline{\pounds 700}}.$$

(e) What choice of vans should be made to minimise the cost if the restriction about the large and small vans is removed? Work out the cost in this case. (2)

Solution



Start at the top-left and move down the line until the last point with integer coordinates is left: $(7, 1)$:

$$C = 80(7) + 60(1) = \underline{\underline{£620}}.$$