Dr Oliver Mathematics
Mathematics
Logarithms Part 3
Past Examination Questions

This booklet consists of 19 questions across a variety of examination topics.
The total number of marks available is 161.
A number of problems are in the integration chapter.

1. Figure 1 shows a sketch of part of the curve with equation (5)

$$y = x\mathrm{e}^{2x},\ x \geqslant 0.$$

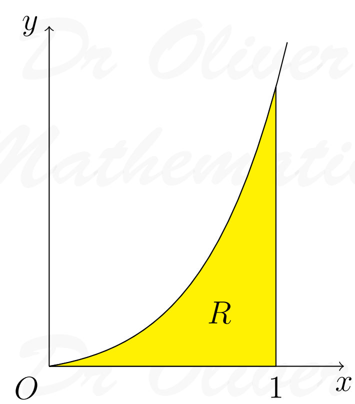

Figure 1: $y = x\mathrm{e}^{2x}$

The finite region R bounded by the lines $x = 1$, the x-axis, and the curve is shown shaded in the figure. Use integration to find the exact value for R.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

Now,

$$\begin{aligned}
\text{area} &= \int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x \\
&= \left[\tfrac{1}{2}x\mathrm{e}^{2x}\right]_{x=0}^{1} - \int_0^1 \tfrac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x \\
&= \left(\tfrac{1}{2}\mathrm{e}^2 - 0\right) - \left[\tfrac{1}{4}\mathrm{e}^{2x}\right]_{x=0}^{1} \\
&= \tfrac{1}{2}\mathrm{e}^2 - \left(\tfrac{1}{4}\mathrm{e}^2 - \tfrac{1}{4}\right) \\
&= \underline{\underline{\tfrac{1}{4}\mathrm{e}^2 + \tfrac{1}{4}}}.
\end{aligned}$$

2. Figure 2 shows a sketch of part of the curve with equation $y = x\mathrm{e}^x$. (5)

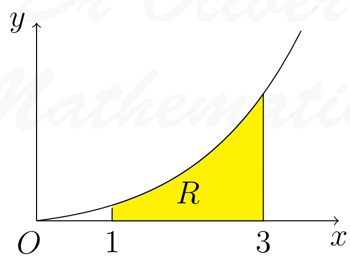

Figure 2: $y = x\mathrm{e}^x$

The finite region R bounded by the line $x = 1$, the line $x = 3$, the x-axis, and the curve is shown shaded in the figure. The region R is rotated through degrees about the x-axis. Use integration by parts to find the exact value for the volume of the solid generated.

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x},$$

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}.$$

Now,

$$\begin{aligned}
\text{volume} &= \pi \int_1^3 x^2 e^{2x}\, dx \\
&= \pi \left\{ \left[\tfrac{1}{2}x^2 e^{2x}\right]_{x=1}^3 - \int_1^3 x e^{2x}\, dx \right\} \\
&= \pi \left\{ \left[\tfrac{1}{2}x^2 e^{2x}\right]_{x=1}^3 - \left[\tfrac{1}{2}x e^{2x}\right]_{x=1}^3 + \int_1^3 \tfrac{1}{2} e^{2x}\, dx \right\} \\
&= \pi \left[\tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}x e^{2x} + \tfrac{1}{4} e^{2x}\right]_{x=1}^3 \\
&= \pi \left[(\tfrac{9}{2}e^6 - \tfrac{3}{2}e^6 + \tfrac{1}{4}e^6) - (0 + 0 + \tfrac{1}{4}e^2) \right] \\
&= \underline{\underline{\pi(\tfrac{13}{4}e^6 - \tfrac{1}{4}e^2)}}.
\end{aligned}$$

3. (a) Given that $y = 2^x$, and using the result $2^x = e^{x \ln 2}$, or otherwise, show that $\dfrac{dy}{dx} = 2^x \ln 2$. (2)

Solution

$$\begin{aligned}
y = 2^x \Rightarrow y &= e^{x \ln 2} \\
\Rightarrow \frac{dy}{dx} &= (\ln 2) e^{x \ln 2} \\
\Rightarrow \underline{\underline{\frac{dy}{dx} = 2^x \ln 2}}.
\end{aligned}$$

(b) Find the gradient of the curve with equation $y = 2^{x^2}$ at the point with coordinates $(2, 16)$. (4)

Solution

$$\begin{aligned}
y = 2^{x^2} &\Rightarrow \ln y = \ln 2^{x^2} \\
&\Rightarrow \ln y = x^2 \ln 2 \\
&\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2x \ln 2 \\
&\Rightarrow \frac{dy}{dx} = 2x(2^{x^2}) \ln 2,
\end{aligned}$$

and

$$x = 2 \Rightarrow \frac{dy}{dx} = 4 \times 2^4 \times \ln 2 = \underline{\underline{64 \ln 2}}.$$

4. A population growth is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,$$

where P is the population, t is the time measured in days, and k is a positive constant. Given that the initial population is P_0,

(a) solve the differential equation, giving P in terms of P_0, k, and t. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} = kP &\Rightarrow \frac{1}{P}\,\mathrm{d}P = k \\ &\Rightarrow \int \frac{1}{P}\,\mathrm{d}P = \int k\,\mathrm{d}t \\ &\Rightarrow \ln P = kt + c \text{ (because } P > 0) \\ &\Rightarrow P = \mathrm{e}^{kt+c} \\ &\Rightarrow P = \mathrm{e}^{c}\mathrm{e}^{kt} \\ &\Rightarrow P = A\mathrm{e}^{kt},\end{aligned}$$

where A is some constant. Now,

$$t = 0 \Rightarrow P_0 \Rightarrow A$$

and we have

$$\underline{\underline{P = P_0\mathrm{e}^{kt}}}.$$

Given also that $k = 2.5$,

(b) find the time taken, to the nearest minute, for the population to reach $2P_0$. (3)

Solution

$$\begin{aligned}2P_0 = P_0\mathrm{e}^{2.5t} &\Rightarrow 2 = \mathrm{e}^{2.5t} \\ &\Rightarrow \ln 2 = 2.5t \\ &\Rightarrow t = \tfrac{2}{5}\ln 2 \\ &\Rightarrow t = 0.277\,258\,872\,2 \text{ days (FCD)} \\ &\Rightarrow t = 6.654\,212\,933 \text{ hours (FCD)} \\ &\Rightarrow t = 399.252\,776 \text{ minutes (FCD)} \\ &\Rightarrow \underline{\underline{t = 399 \text{ minutes (nearest minute)}}}.\end{aligned}$$

In an improved model, the differential equation is given as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days, and λ is a positive constant. Given, again, that the initial population is P_0 and that the time measured in days,

(c) solve the second differential equation, giving P in terms of P_0, λ, and t. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \cos \lambda t &\Rightarrow \frac{1}{P}\,\mathrm{d}P = \lambda \cos \lambda t \\ &\Rightarrow \int \frac{1}{P}\,\mathrm{d}P = \int \lambda \cos \lambda t \,\mathrm{d}t \\ &\Rightarrow \ln P = \sin \lambda t + c \text{ (because } P > 0) \\ &\Rightarrow P = \mathrm{e}^{\sin \lambda t + c} \\ &\Rightarrow P = \mathrm{e}^{c}\mathrm{e}^{\sin \lambda t} \\ &\Rightarrow P = A\mathrm{e}^{\sin \lambda t},\end{aligned}$$

where A is some constant. Now,

$$t = 0 \Rightarrow P_0 \Rightarrow A$$

and we have

$$\underline{\underline{P = P_0 \mathrm{e}^{\sin \lambda t}}}.$$

Given also that $\lambda = 2.5$,

(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. (3)

Solution

$$
\begin{aligned}
2P_0 = P_0 \mathrm{e}^{\sin 2.5t} &\Rightarrow 2 = \mathrm{e}^{\sin 2.5t} \\
&\Rightarrow \ln 2 = \sin 2.5t \\
&\Rightarrow 2.5t = \arcsin(\ln 2) \\
&\Rightarrow t = \tfrac{2}{5}\arcsin(\ln 2) \\
&\Rightarrow t = 0.306\,338\,477\,9 \text{ days (FCD)} \\
&\Rightarrow t = 7.352\,123\,47 \text{ hours (FCD)} \\
&\Rightarrow t = 441.127\,408\,2 \text{ minutes (FCD)} \\
&\Rightarrow \underline{\underline{t = 441 \text{ minutes (nearest minute)}}}.
\end{aligned}
$$

5. The curve C has the equation $y\mathrm{e}^{-2x} = 2x + y^2$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. (5)

Solution

$$
\begin{aligned}
& \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{-2x} - 2y\mathrm{e}^{-2x} = 2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{-2x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 2y\mathrm{e}^{-2x} \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x}(\mathrm{e}^{-2x} - 2y) = 2 + 2y\mathrm{e}^{-2x} \\
\Rightarrow\ & \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 + 2y\mathrm{e}^{-2x}}{\mathrm{e}^{-2x} - 2y}}}.
\end{aligned}
$$

The point P on C has coordinates $(0, 1)$.

(b) Find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b, and c are integers. (4)

Solution

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+2}{1-2} = -4 \Rightarrow m' = \frac{1}{4}$$

and

$$y - 1 = \tfrac{1}{4}(x - 0) \Rightarrow 4y - 4 = x \Rightarrow \underline{\underline{x - 4y + 4 = 0}}.$$

6. (a) Find $\displaystyle\int \frac{9x + 6}{x}\,\mathrm{d}x$, $x > 0$. (2)

Solution

$$\int \frac{9x+6}{x}\,\mathrm{d}x = \int \left(9 + \frac{6}{x}\right)\,\mathrm{d}x$$
$$= \underline{\underline{9x + 6\ln x + c}},$$

because we know that $x > 0$.

(b) Given that $y = 8$ at $x = 1$, solve the differential equation (6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(9x+6)y^{\frac{1}{3}}}{x},$$

giving your answer in the form $y^2 = \mathrm{g}(x)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(9x+6)y^{\frac{1}{3}}}{x} \Rightarrow \frac{1}{y^{\frac{1}{3}}}\,\mathrm{d}y = \left(9 + \frac{6}{x}\right)\,\mathrm{d}x$$
$$\Rightarrow y^{-\frac{1}{3}}\,\mathrm{d}y = \left(9 + \frac{6}{x}\right)\,\mathrm{d}x$$
$$\Rightarrow \int y^{-\frac{1}{3}}\,\mathrm{d}y = \int \left(9 + \frac{6}{x}\right)\,\mathrm{d}x$$
$$\Rightarrow \tfrac{3}{2}y^{\frac{2}{3}} = 9x + 6\ln x + c.$$

Now, $x = 1$ when $y = 8$ and so

$$6 = 9 + 0 + c \Rightarrow c = -3.$$

Hence,

$$\tfrac{3}{2}y^{\frac{2}{3}} = 9x + 6\ln x - 3 \Rightarrow y^{\frac{2}{3}} = \tfrac{2}{3}(9x + 6\ln x - 3)$$
$$\Rightarrow y^{\frac{2}{3}} = 6x + 4\ln x - 2$$
$$\Rightarrow \underline{\underline{y^2 = (6x + 4\ln x - 2)^3}}.$$

7. A curve C has equation (7)

$$2^x + y^2 = 2xy.$$

Find the exact value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on C with coordinates $(3, 2)$.

Solution

$$
\begin{aligned}
2^x + y^2 = 2xy &\Rightarrow (\ln 2)2^x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} \\
&\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - (\ln 2)2^x \\
&\Rightarrow (2y - 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - (\ln 2)2^x \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y - (\ln 2)2^x}{2y - 2x}.
\end{aligned}
$$

Now,

$$x = 3, y = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 - 8\ln 2}{4 - 6} = \underline{\underline{4\ln 2 - 2}}.$$

8. The current, I amps, in an electric circuit at time t seconds is given by (5)

$$I = 16 - 16(0.5)^t,\ t \geqslant 0.$$

Use differentiation to find the value of $\frac{\mathrm{d}I}{\mathrm{d}t}$ when $t = 3$. Give your answer in the form $\ln a$, where a is a constant.

Solution

$$I = 16 - 16(0.5)^t \Rightarrow \frac{\mathrm{d}I}{\mathrm{d}t} = -16(\ln 0.5)(0.5)^t$$

and so

$$t = 3 \Rightarrow \frac{\mathrm{d}I}{\mathrm{d}t} = -16(\ln 0.5)(0.5)^3 = -2(\ln 0.5) = -\ln 0.25 = \underline{\underline{\ln 4}}.$$

9. Find the gradient of the curve with equation (7)

$$\ln y = 2x\ln x,\ x > 0,\ y > 0,$$

at the point on the curve where $x = 2$. Give your answer an an exact value.

Solution

$$\begin{aligned}\ln y = 2x\ln x &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2\times\ln x + 2x\times\frac{1}{x}\\ &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln x + 2\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y(2\ln x + 2).\end{aligned}$$

Now,

$$\begin{aligned}\ln y = 4\ln 2 &\Rightarrow \ln y = \ln 2^4\\ &\Rightarrow \ln y = \ln 16\\ &\Rightarrow y = 16\end{aligned}$$

and

$$\underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 16(2\ln 2 + 2).}}$$

10. (a) Express $\dfrac{1}{P(5-P)}$ in partial fractions. (2)

Solution

$$\frac{1}{P(5-P)} \equiv \frac{A}{P} + \frac{B}{5-P} \equiv \frac{A(5-P)+BP}{P(5-P)}$$

and so

$$1 \equiv A(5-P) + BP.$$

Now,
$\underline{P=0}$: $1 = 5A \Rightarrow A = \frac{1}{5}$.
$\underline{P=5}$: $1 = 5B \Rightarrow B = \frac{1}{5}$.
Hence,

$$\frac{1}{P(5-P)} \equiv \underline{\underline{\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{5-P}}}.$$

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \tfrac{1}{15}P(5 - P),\ t \geqslant 0,$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began. Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, given your answer in the form (8)

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}},$$

where a, b, and c are integers.

Solution

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} = \tfrac{1}{15}P(5-P) &\Rightarrow \frac{1}{P(5-P)}\,\mathrm{d}P = \tfrac{1}{15}\,\mathrm{d}t \\
&\Rightarrow \left(\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{5-P}\right)\mathrm{d}P = \tfrac{1}{15}\,\mathrm{d}t \\
&\Rightarrow \left(\frac{1}{P} + \frac{1}{5-P}\right)\mathrm{d}P = \tfrac{1}{3}\,\mathrm{d}t \\
&\Rightarrow \int \left(\frac{1}{P} + \frac{1}{5-P}\right)\mathrm{d}P = \int \tfrac{1}{3}\,\mathrm{d}t \\
&\Rightarrow \ln P - \ln(5-P) = \tfrac{1}{3}t + c \\
&\Rightarrow \ln\left(\frac{P}{5-P}\right) = \tfrac{1}{3}t + c.
\end{aligned}$$

Now, $t = 0$ as $P = 1$:

$$\ln\left(\frac{1}{5-1}\right) = \tfrac{1}{3} \times 0 + c \Rightarrow \ln \tfrac{1}{4} = c,$$

and

$$\begin{aligned}\ln\left(\frac{P}{5-P}\right) = \tfrac{1}{3}t + \ln\tfrac{1}{4} &\Rightarrow \ln\left(\frac{P}{5-P}\right) - \ln\tfrac{1}{4} = \tfrac{1}{3}t \\ &\Rightarrow \ln\left(\frac{4P}{5-P}\right) = \tfrac{1}{3}t \\ &\Rightarrow \frac{4P}{5-P} = \mathrm{e}^{\frac{1}{3}t} \\ &\Rightarrow 4P = \mathrm{e}^{\frac{1}{3}t}(5-P) \\ &\Rightarrow 4P = 5\mathrm{e}^{\frac{1}{3}t} - P\mathrm{e}^{\frac{1}{3}t} \\ &\Rightarrow 4P + P\mathrm{e}^{\frac{1}{3}t} = 5\mathrm{e}^{\frac{1}{3}t} \\ &\Rightarrow P(4 + \mathrm{e}^{\frac{1}{3}t}) = 5\mathrm{e}^{\frac{1}{3}t} \\ &\Rightarrow P = \frac{5\mathrm{e}^{\frac{1}{3}t}}{4 + \mathrm{e}^{\frac{1}{3}t}} \\ &\Rightarrow \underline{\underline{P = \frac{5}{1 + 4\mathrm{e}^{-\frac{1}{3}t}}}}.\end{aligned}$$

(c) Hence show that the population cannot exceed 5000. (1)

Solution
As $t \to \infty$, $P \to \frac{5}{1+0} = 5$ and so the population <u>cannot exceed 5000</u>.

11. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3°C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ°C. The rate of the change of the temperature of the water in the bottle is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{3-\theta}{125}.$$

(a) By solving the differential equation, show that (4)

$$\theta = A\mathrm{e}^{-0.008t} + 3,$$

where A is a constant.

Solution

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{3-\theta}{125} &\Rightarrow \frac{1}{3-\theta}\,\mathrm{d}\theta = 0.008\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{3-\theta}\,\mathrm{d}\theta = \int 0.008\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{\theta-3}\,\mathrm{d}\theta = -\int 0.008\,\mathrm{d}t \\ &\Rightarrow \ln(\theta-3) = -0.008t + c \\ &\Rightarrow \theta - 3 = \mathrm{e}^{-0.008t+c} \\ &\Rightarrow \theta = \mathrm{e}^{-0.008t}\,\mathrm{e}^{c} + 3 \\ &\Rightarrow \underline{\underline{\theta = A\mathrm{e}^{-0.008t} + 3}}.\end{aligned}$$

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16°C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10°C, giving your answer to the nearest minute. (5)

Solution

Now, $t = 0$ when $\theta = 16$:

$$\ln(16-3) = -0.008 \times 0 + c \Rightarrow c = \ln 13,$$

and so

$$\begin{aligned}\ln(\theta-3) = -0.008t + \ln 13 &\Rightarrow \ln(3-\theta) - \ln 13 = -0.008t \\ &\Rightarrow \ln\left(\frac{3-\theta}{13}\right) = -0.008t \\ &\Rightarrow \frac{3-\theta}{13} = \mathrm{e}^{-0.008t} \\ &\Rightarrow \theta - 3 = 13\mathrm{e}^{-0.008t} \\ &\Rightarrow \theta = 13\mathrm{e}^{-0.008t} + 3.\end{aligned}$$

So,

$$\begin{aligned}10 = 13\mathrm{e}^{-0.008t} + 3 &\Rightarrow 7 = 13\mathrm{e}^{-0.008t} \\ &\Rightarrow \tfrac{7}{13} = \mathrm{e}^{-0.008t} \\ &\Rightarrow \tfrac{13}{7} = \mathrm{e}^{0.008t} \\ &\Rightarrow 0.008t = \ln \tfrac{13}{7} \\ &\Rightarrow t = 125 \ln \tfrac{13}{7} \\ &\Rightarrow t = 77.379\,901\,05 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 77 \text{ (nearest minute)}}}.\end{aligned}$$

12. Water is being heated in a kettle. At time t seconds, the temperature of the water is θ°C. The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(120 - \lambda),\ \lambda \leqslant 100,$$

where λ is a positive constant. Given that $\lambda = 20$ when $t = 0$,

(a) solve this differential equation to show that (8)

$$\lambda = 120 - 100\mathrm{e}^{-\lambda t}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(120 - \theta) &\Rightarrow \frac{1}{120 - \theta}\,\mathrm{d}\theta = \lambda\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{120 - \theta}\,\mathrm{d}\theta = \int \lambda\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{\theta - 120}\,\mathrm{d}\theta = -\int \lambda\,\mathrm{d}t \\ &\Rightarrow \ln(\theta - 120) = -\lambda t + c \\ &\Rightarrow \theta - 120 = \mathrm{e}^{-\lambda t + c} \\ &\Rightarrow \theta - 120 = \mathrm{e}^{-\lambda t}\,\mathrm{e}^{c} \\ &\Rightarrow \theta - 120 = k\mathrm{e}^{-\lambda t} \\ &\Rightarrow \theta = 120 + k\mathrm{e}^{-\lambda t}.\end{aligned}$$

Now, $\lambda = 20$ when $t = 0$,

$$20 = 120 + k \Rightarrow k = -100$$

we have

$$\underline{\underline{\theta = 120 - 100\mathrm{e}^{-\lambda t}}}.$$

When the temperature of the water reaches 100°C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off. (3)

Solution

$$\begin{aligned}100 = 120 - 100\mathrm{e}^{-0.01t} &\Rightarrow -20 = -100\mathrm{e}^{-0.01t}\\ &\Rightarrow \mathrm{e}^{-0.01t} = \tfrac{1}{5}\\ &\Rightarrow -0.01t = \ln\tfrac{1}{5}\\ &\Rightarrow t = -100\ln\tfrac{1}{5}\\ &\Rightarrow t = 160.943\,791\,2 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{t = 161 \text{ (nearest second)}}}.\end{aligned}$$

13. The curve C has equation (7)

$$3^{x-1} + xy - y^2 + 5 = 0.$$

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 3)$ on the curve C can be written in the form $\dfrac{1}{\lambda}\ln(\mu\mathrm{e}^3)$, where λ and μ are integers to be found.

Solution

$$\begin{aligned}3^{x-1} + xy - y^2 + 5 = 0 &\Rightarrow \ln 3(3^{x-1}) + y + x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 0 = 0\\ &\Rightarrow \ln 3(3^{x-1}) + y = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x}\\ &\Rightarrow \ln 3(3^{x-1}) + y = \frac{\mathrm{d}y}{\mathrm{d}x}(2y - x)\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 3(3^{x-1}) + y}{2y - x}.\end{aligned}$$

Now, $x = 1$ when $y = 3$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\ln 3(3^0) + 3}{6 - 1} \\ &= \tfrac{1}{5}(\ln 3 + 3) \\ &= \tfrac{1}{5}(\ln 3 + \ln \mathrm{e}^3) \\ &= \underline{\underline{\tfrac{1}{5}\ln(3\mathrm{e}^3)}}.\end{aligned}$$

14. In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t after the start of the experiment. It is known that at time t, the rate of increase of the mass of waste products, in kg per minute, is k times of the mass of unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(M - x),$$

where M is a constant.

(a) Explain, in the context of the problem, what $\frac{\mathrm{d}x}{\mathrm{d}t}$ and M represent. (2)

Solution
$\frac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of increase of the mass of waste products;
M is the total mass of unburned fuel and waste fuel.

Given that initially the mass of waste products is zero,

(b) solve the differential equation, expressing x in terms of k, M, and t. (6)

Solution

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = k(M-x) &\Rightarrow \frac{1}{M-x}\,\mathrm{d}x = k\,\mathrm{d}t \\ &\Rightarrow \frac{1}{x-M}\,\mathrm{d}x = -k\,\mathrm{d}t \\ &\Rightarrow \int \frac{1}{x-M}\,\mathrm{d}x = -\int k\,\mathrm{d}t \\ &\Rightarrow \ln(x-M) = -kt + c \\ &\Rightarrow x - M = \mathrm{e}^{-kt+c} \\ &\Rightarrow x - M = \mathrm{e}^{-kt}\,\mathrm{e}^{c} \\ &\Rightarrow x - M = d\mathrm{e}^{-kt} \\ &\Rightarrow x = M + d\mathrm{e}^{-kt}.\end{aligned}$$

Now, $x = 0$ when $t = 0$:

$$0 = M + d \Rightarrow d = -M$$

and

$$\underline{\underline{x = M - M\mathrm{e}^{-kt}}}.$$

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(c) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form. (4)

Solution

$x = \frac{1}{2}M$ when $t = \ln 4$:

$$\begin{aligned}\tfrac{1}{2}M = M - M\mathrm{e}^{-k\ln 4} &\Rightarrow -\tfrac{1}{2}M = -M\mathrm{e}^{\ln 4^{-k}} \\ &\Rightarrow \tfrac{1}{2} = 4^{-k} \\ &\Rightarrow k = \tfrac{1}{2}\end{aligned}$$

and so

$$x = M - M\mathrm{e}^{-\frac{1}{2}t}$$

and $t = \ln 9$ gives

$$x = M - M\mathrm{e}^{-\frac{1}{2}\ln 9} = M - M\mathrm{e}^{\ln 9^{-\frac{1}{2}}} = M - \tfrac{1}{3}M = \underline{\underline{\tfrac{2}{3}M}}.$$

15. Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \operatorname{cosec} 2y \operatorname{cosec} y.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \operatorname{cosec} 2y \operatorname{cosec} y &\Rightarrow \sin 2y \sin y \, \mathrm{d}y = \mathrm{e}^x \, \mathrm{d}x \\ &\Rightarrow 2\sin^2 y \cos y \, \mathrm{d}y = \mathrm{e}^x \, \mathrm{d}x \\ &\Rightarrow \int 2\sin^2 y \cos y \, \mathrm{d}y = \int \mathrm{e}^x \, \mathrm{d}x \\ &\Rightarrow \tfrac{2}{3}\sin^3 y = \mathrm{e}^x + c.\end{aligned}$$

Now, $y = \frac{\pi}{6}$ at $x = 0$:

$$\tfrac{2}{3}\sin^3 \tfrac{\pi}{6} = 1 + c \Rightarrow \tfrac{1}{12} = 1 + c \Rightarrow c = -\tfrac{11}{12}$$

and we have

$$\underline{\underline{\tfrac{2}{3}\sin^3 y = \mathrm{e}^x - \tfrac{11}{12}}}.$$

16. The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{(kt-1)(5000-N)}{t}, \; t > 0, \; 0 < N < 5000.$$

In a given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(a) By solving the differential equation, show that (5)

$$N = 5000 - At\mathrm{e}^{-kt},$$

where A is a positive constant.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{(kt-1)(5000-N)}{t} &\Rightarrow \frac{1}{5000-N}\,\mathrm{d}N = \frac{kt-1}{t}\,\mathrm{d}t \\
&\Rightarrow \frac{1}{5000-N}\,\mathrm{d}N = \left(k - \frac{1}{t}\right)\,\mathrm{d}t \\
&\Rightarrow \int \frac{1}{5000-N}\,\mathrm{d}N = \int \left(k - \frac{1}{t}\right)\,\mathrm{d}t \\
&\Rightarrow -\ln(5000-N) = kt - \ln t + c \\
&\Rightarrow \ln t - \ln(5000-N) = kt + c \\
&\Rightarrow \ln\left(\frac{t}{5000-N}\right) = kt + c \\
&\Rightarrow \frac{t}{5000-N} = \mathrm{e}^{kt+c} \\
&\Rightarrow 5000 - N = t\mathrm{e}^{-kt-c} \\
&\Rightarrow 5000 - N = t\mathrm{e}^{-kt}\,\mathrm{e}^{-c} \\
&\Rightarrow \underline{\underline{N = 5000 - At\mathrm{e}^{-kt}}}.
\end{aligned}
$$

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(b) Find the exact value of the constant A and the exact value of the constant k. (4)

Solution

$$
\begin{aligned}
1800 = 5000 - 2A\mathrm{e}^{-2k} &\Rightarrow 2A\mathrm{e}^{-2k} = 3200 \\
1200 = 5000 - A\mathrm{e}^{-k} &\Rightarrow A\mathrm{e}^{-k} = 3800.
\end{aligned}
$$

Now,

$$
\begin{aligned}
A\mathrm{e}^{-k} = 3800 &\Rightarrow A = 3800\mathrm{e}^{k} \\
&\Rightarrow 2(3800\mathrm{e}^{k})\mathrm{e}^{-2k} = 3200 \\
&\Rightarrow 7600\mathrm{e}^{-k} = 3200 \\
&\Rightarrow \tfrac{19}{8} = \mathrm{e}^{k} \\
&\Rightarrow \underline{\underline{k = \ln \tfrac{19}{8}}} \\
&\Rightarrow A = 3800\mathrm{e}^{\ln \frac{19}{8}} \\
&\Rightarrow \underline{\underline{A = 9025}}.
\end{aligned}
$$

(c) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish. (1)

Solution

$$N = 5000 - 45125\mathrm{e}^{-5\ln\frac{19}{8}} = 4402.828\,401 \text{ (FCD)} = \underline{\underline{4400 \text{ (nearest 100)}}}.$$

17. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions. (2)

Solution

$$\frac{2}{P(P-2)} \equiv \frac{A}{P} + \frac{B}{P-2} \equiv \frac{A(P-2)+BP}{P(P-2)}$$

and so

$$2 \equiv A(P-2) + BP.$$

Now,
$\underline{P=0}$: $2 = -2A \Rightarrow A = -1$.
$\underline{P=2}$: $2 = 2B \Rightarrow B = 1$.
Hence,

$$\frac{2}{P(P-2)} \equiv \underline{\underline{\frac{1}{P-2} - \frac{1}{P}}}.$$

A team of biologists is studying a population of a particular species of animal.
The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \tfrac{1}{2}P(P-2)\cos 2t,\ t \geqslant 0,$$

where P is the population in thousands and t is the time measured in years since the start of the study.
Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that (7)

$$P = \frac{6}{3 - \mathrm{e}^{\frac{1}{2}\sin 2t}},$$

Solution

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} = \tfrac{1}{2}P(P-2)\cos 2t \Rightarrow \frac{2}{P(P-2)}\,\frac{\mathrm{d}P}{\mathrm{d}t} &= \cos 2t\,\mathrm{d}t \\
\Rightarrow \left(\frac{1}{P-2} - \frac{1}{P}\right)\frac{\mathrm{d}P}{\mathrm{d}t} &= \cos 2t\,\mathrm{d}t \\
\Rightarrow \int \left(\frac{1}{P-2} - \frac{1}{P}\right)\mathrm{d}P &= \int \cos 2t\,\mathrm{d}t \\
\Rightarrow \ln(P-2) - \ln P &= \tfrac{1}{2}\sin 2t + c.
\end{aligned}
$$

Now, $P = 3$ when $t = 0$:

$$0 - \ln 3 - 0 = 0 + c \Rightarrow c = -\ln 3$$

and

$$
\begin{aligned}
\ln(P-2) - \ln P = \tfrac{1}{2}\sin 2t - \ln 3 \Rightarrow \ln\left(\frac{P-2}{P}\right) &= \tfrac{1}{2}\sin 2t - \ln 3 \\
\Rightarrow \frac{P-2}{P} &= \mathrm{e}^{\frac{1}{2}\sin 2t - \ln 3} \\
\Rightarrow \frac{P-2}{P} &= \mathrm{e}^{\frac{1}{2}\sin 2t}\,\mathrm{e}^{-\ln 3} \\
\Rightarrow \frac{P-2}{P} &= \tfrac{1}{3}\mathrm{e}^{\frac{1}{2}\sin 2t} \\
\Rightarrow P - 2 &= \tfrac{1}{3}P\mathrm{e}^{\frac{1}{2}\sin 2t} \\
\Rightarrow P - \tfrac{1}{3}P\mathrm{e}^{\frac{1}{2}\sin 2t} &= 2 \\
\Rightarrow P(1 - \tfrac{1}{3}\mathrm{e}^{\frac{1}{2}\sin 2t}) &= 2 \\
\Rightarrow P &= \frac{2}{1 - \frac{1}{3}\mathrm{e}^{\frac{1}{2}\sin 2t}} \\
\Rightarrow P &= \underline{\underline{\frac{6}{3 - \mathrm{e}^{\frac{1}{2}\sin 2t}}}}.
\end{aligned}
$$

(c) find the time taken for the population to reach 4000 for the first time. Give your answer in years to 3 significant figures. (3)

Solution

$$\begin{aligned}4 = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}} &\Rightarrow 12 - 4e^{\frac{1}{2}\sin 2t} = 6\\ &\Rightarrow 4e^{\frac{1}{2}\sin 2t} = 6\\ &\Rightarrow e^{\frac{1}{2}\sin 2t} = \tfrac{3}{2}\\ &\Rightarrow \tfrac{1}{2}\sin 2t = \ln\tfrac{3}{2}\\ &\Rightarrow \sin 2t = 2\ln\tfrac{3}{2}\\ &\Rightarrow 2t = \arcsin(2\ln\tfrac{3}{2})\\ &\Rightarrow t = \tfrac{1}{2}\arcsin(2\ln\tfrac{3}{2})\\ &\Rightarrow t = 0.472\,870\,046\,7 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{t = 0.473 \text{ (3 sf)}}}.\end{aligned}$$

18. The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{dx}{dt} = -\tfrac{5}{2}x,\ t \geqslant 0,$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(a) solve the differential equation, giving x in terms of t. You should show all of the steps in your working and give your answer in its simplest form. (4)

Solution

$$\begin{aligned}\frac{dx}{dt} = -\tfrac{5}{2}x &\Rightarrow \frac{1}{x}\,dx = -\tfrac{5}{2}\,dt\\ &\Rightarrow \int \frac{1}{x}\,dx = -\int \tfrac{5}{2}\,dt\\ &\Rightarrow \ln x = -\tfrac{5}{2}t + c.\end{aligned}$$

Now, $x = 60$ when $t = 0$:

$$\ln 60 = 0 + c \Rightarrow c = \ln 60$$

and so

$$\begin{aligned}\ln x = -\tfrac{5}{2}t + \ln 60 &\Rightarrow x = \mathrm{e}^{-\frac{5}{2}t + \ln 60} \\ &\Rightarrow x = \mathrm{e}^{-\frac{5}{2}t}\,\mathrm{e}^{\ln 60} \\ &\Rightarrow \underline{\underline{x = 60\mathrm{e}^{-\frac{5}{2}t}}}.\end{aligned}$$

(b) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams. Give your answer to the nearest minute. (3)

Solution

$$\begin{aligned}20 = 60\mathrm{e}^{-\frac{5}{2}t} &\Rightarrow \mathrm{e}^{-\frac{5}{2}t} = \tfrac{1}{3} \\ &\Rightarrow -\tfrac{5}{2}t = \ln\tfrac{1}{3} \\ &\Rightarrow t = -\tfrac{2}{5}\ln\tfrac{1}{3} \\ &\Rightarrow t = 0.439\,444\,915\,5 \text{ days (FCD)} \\ &\Rightarrow t = 10.546\,677\,97 \text{ hours (FCD)} \\ &\Rightarrow t = 632.800\,678\,3 \text{ minutes (FCD)} \\ &\Rightarrow t = 633 \text{ minutes (nearest minute).}\end{aligned}$$

19. The curve C has equation

$$4x^2 - y^3 - 4xy + 2^y = 0.$$

The point P with coordinates $(-2, 4)$ lies on C.

(a) Find the exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point P. (6)

Solution

$$\begin{aligned}4x^2 - y^3 - 4xy + 2^y = 0 &\Rightarrow 8x - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 4y - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + (\ln 2)2^y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow 8x - 4y = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} - (\ln 2)2^y\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow 8x - 4y = \frac{\mathrm{d}y}{\mathrm{d}x}(3y^2 + 4x - (\ln 2)2^y) \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8x - 4y}{3y^2 + 4x - (\ln 2)2^y}\end{aligned}$$

and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} = \frac{-16-16}{48-8-16\ln 2} = \frac{-32}{40-16\ln 2} = \underline{\underline{\frac{4}{2\ln 2-5}}}.$$

The normal to C at P meets the y-axis at the point A.

(b) Find the y-coordinate of A, giving your answer in the form $p + q\ln 2$, where p and q are constants to be determined. (3)

Solution

$$\begin{aligned} y = -\frac{2\ln 2-5}{4}x + c &\Rightarrow 4 = \frac{2(2\ln 2-5)}{4} + c \\ &\Rightarrow 4 = \frac{2\ln 2-5}{2} + c \\ &\Rightarrow c = 4 - \frac{2\ln 2-5}{2} \\ &\Rightarrow c = \frac{8-(2\ln 2-5)}{2} \\ &\Rightarrow c = \frac{13-2\ln 2}{2} \\ &\Rightarrow \underline{\underline{c = \tfrac{13}{2} - \ln 2}}. \end{aligned}$$