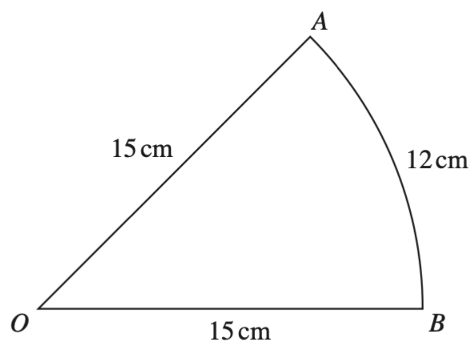
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2009 June Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The diagram shows a sector AOB of a circle, centre O, radius 15 cm.



The length of the arc AB is 12 cm.

(a) Find, in radians, angle AOB. (2)

Solution

$$15 \times \angle AOB = 12 \Rightarrow \underline{\underline{\angle AOB = \tfrac{4}{5}}}.$$

(b) Find the area of the sector AOB. (2)

Solution

Well,

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \times 15^2 \times \tfrac{4}{5} \\ &= \underline{\underline{90 \text{ cm}^2}}. \end{aligned}$$

2. The equation of a curve is (4)

$$y = x^3 - 8.$$

Find the equation of the normal to the curve at the point where the curve crosses the x-axis.

Solution

Now,

$$\begin{aligned} x^3 - 8 = 0 &\Rightarrow x^3 = 8 \\ &\Rightarrow x = 2, \end{aligned}$$

and so the point where the curve crosses the x-axis is $(2, 0)$.

Next,

$$y = x^3 - 8 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$$

and

$$\begin{aligned} x = 2 &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12 \\ &\Rightarrow m_{\text{normal}} = -\tfrac{1}{12}. \end{aligned}$$

Finally, the equation of the normal is

$$y - 0 = -\tfrac{1}{12}(x - 2) \Rightarrow \underline{\underline{y = -\tfrac{1}{12}x + \tfrac{1}{6}}}.$$

3. Show that (4)

$$\frac{1 - \cos^2\theta}{\sec^2\theta - 1} \equiv 1 - \sin^2\theta.$$

Solution

$$\begin{aligned} \frac{1 - \cos^2\theta}{\sec^2\theta - 1} &\equiv \frac{\sin^2\theta}{\tan^2\theta} \\ &\equiv \cos^2\theta \\ &\equiv \underline{\underline{1 - \sin^2\theta}}, \end{aligned}$$

as required.

4. The line

$$y = 5x - 3$$

is a tangent to the curve

$$y = kx^2 - 3x + 5$$

at the point A.

Find

(a) the value of k, (3)

Solution

Well,

$$kx^2 - 3x + 5 = 5x - 3 \Rightarrow kx^2 - 8x + 8 = 0$$

and

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (-8)^2 - 4(k)(8) = 0 \\ &\Rightarrow 32k = 64 \\ &\Rightarrow \underline{\underline{k = 2}}. \end{aligned}$$

(b) the coordinates of A. (2)

Solution

$$2x^2 - 8x + 8 = 0 \Rightarrow 2(x^2 - 4x + 4) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +4 \end{array}\right\} -2, -2$$

$$\begin{aligned} &\Rightarrow 2(x-2)^2 = 0 \\ &\Rightarrow x - 2 = 0 \\ &\Rightarrow x = 2 \\ &\Rightarrow y = 7; \end{aligned}$$

hence, $\underline{\underline{A(2, 7)}}$.

5. (a) Solve the equation (3)

$$9^{2x-1} = 27^x.$$

Solution

$$\begin{aligned} 9^{2x-1} = 27^x &\Rightarrow (3^2)^{2x-1} = (3^3)^x \\ &\Rightarrow 3^{2(2x-1)} = 3^{3x} \end{aligned}$$

take the logarithms to the base 3:

$$\begin{aligned} &\Rightarrow 2(2x-1) = 3x \\ &\Rightarrow 4x - 2 = 3x \\ &\Rightarrow \underline{\underline{x = 2}}. \end{aligned}$$

(b) Given that (2)

$$\frac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{\sqrt{a^3b^{-\frac{2}{3}}}} = a^p b^q,$$

find the value of p and of q.

Solution

$$\begin{aligned} \frac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{\sqrt{a^3b^{-\frac{2}{3}}}} &= \frac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{a^{\frac{3}{2}}b^{-\frac{1}{3}}} \\ &= a^{-2}b^1; \end{aligned}$$

so, $\underline{\underline{p = -2}}$ and $\underline{\underline{q = 1}}$.

6. Solve the equation (6)

$$2x^3 + 3x^2 - 32x + 15 = 0.$$

Solution

Let

$$\mathrm{f}(x) = 2x^3 + 3x^2 - 32x + 15.$$

Then

$$\begin{aligned}
f(1) &= 2 + 3 - 32 + 15 = -12 \\
f(-1) &= -2 + 3 + 32 + 15 = 48 \\
f(3) &= 54 + 27 - 96 + 15 = 0
\end{aligned}$$

and so $(x - 3)$ is a root.

We use synthetic division:

$$\begin{array}{c|cccc}
3 & 2 & 3 & -32 & 15 \\
 & \downarrow & 6 & 27 & 15 \\
\hline
 & 2 & 9 & -5 & 0
\end{array}$$

so

$$2x^3 + 3x^2 - 32x + 15 = (x - 3)(2x^2 + 9x - 5).$$

Now,

$$\left.\begin{array}{ll} \text{add to:} & +9 \\ \text{multiply to:} & (+2) \times (-5) = -10 \end{array}\right\} -1, +10$$

E.g.,

$$\begin{aligned}
2x^2 + 9x - 5 = 0 &\Rightarrow 2x^2 - x + 10x - 5 = 0 \\
&\Rightarrow x(2x - 1) + 5(2x - 1) = 0 \\
&\Rightarrow (x + 5)(2x - 1) = 0 \\
&\Rightarrow x = -5 \text{ or } x = \tfrac{1}{2}.
\end{aligned}$$

Hence, the solutions are

$$\underline{\underline{x = -5,\ x = \tfrac{1}{2},\ \text{or } x = 3.}}$$

7. (a) Find (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{3x} - \tfrac{1}{3}\mathrm{e}^{3x}).$$

Solution

Product rule:

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \mathrm{e}^{3x} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 3\mathrm{e}^{3x}$$

so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{3x} - \tfrac{1}{3}\mathrm{e}^{3x}) &= (x)(3\mathrm{e}^{3x}) + (1)(\mathrm{e}^{3x}) - \mathrm{e}^{3x} \\ &= 3x\mathrm{e}^{3x} + \mathrm{e}^{3x}) - \mathrm{e}^{3x} \\ &= \underline{\underline{3x\mathrm{e}^{3x}}}.\end{aligned}$$

(b) Hence find (3)

$$\int x\mathrm{e}^{3x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int x\mathrm{e}^{3x}\,\mathrm{d}x &= \tfrac{1}{3}\int 3x\mathrm{e}^{3x}\,\mathrm{d}x \\ &= \tfrac{1}{3}(x\mathrm{e}^{3x} - \tfrac{1}{3}\mathrm{e}^{3x}) + c \\ &= \underline{\underline{\tfrac{1}{3}x\mathrm{e}^{3x} - \tfrac{1}{9}\mathrm{e}^{3x} + c}}.\end{aligned}$$

8. A curve has equation

$$y = \frac{2x}{x^2 + 9}.$$

(a) Find the x-coordinate of each of the stationary points of the curve. (4)

Solution

Quotient rule:

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = x^2 + 9 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2+9)(2)-(2x)(2x)}{(x^2+9)^2}\\ &= \frac{2x^2+18-4x^2}{(x^2+9)^2}\\ &= \frac{18-2x^2}{(x^2+9)^2}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{18-2x^2}{(x^2+9)^2} = 0\\ &\Rightarrow 18-2x^2 = 0\\ &\Rightarrow 2x^2 = 18\\ &\Rightarrow x^2 = 9\\ &\Rightarrow \underline{\underline{x=-3 \text{ or } x=3}}.\end{aligned}$$

(b) Given that x is increasing at the rate of 2 units per second, find the rate of increase of y when $x = 1$. (3)

Solution

Well,

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{4}{25}$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} &\Rightarrow \tfrac{4}{25} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{2}\\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}t} = \tfrac{8}{25}}}.\end{aligned}$$

9. At 10 00 hours, a ship P leaves a point A with position vector $(-4\mathbf{i}+8\mathbf{j})$ km relative to an origin O, where $\mathbf{i}$ is a unit vector due East and $\mathbf{j}$ is a unit vector due North.

The ship sails north-east with a speed of $10\sqrt{2}$ kmh^{-1}.

Find

(a) the velocity vector of P, (2)

Solution
Well,

$$\begin{aligned}\mathbf{v} &= (10\sqrt{2}\cos 45^\circ)\mathbf{i} + (10\sqrt{2}\sin 45^\circ)\mathbf{j} \\ &= \underline{\underline{(10\mathbf{i} + 10\mathbf{j})\ \text{kmh}^{-1}}}.\end{aligned}$$

(b) the position vector of P at 12 00 hours. (2)

Solution

$$\begin{aligned}\overrightarrow{OP} &= (-4\mathbf{i} + 8\mathbf{j}) + 2(10\mathbf{i} + 10\mathbf{j}) \\ &= \underline{\underline{(16\mathbf{i} + 28\mathbf{j})\ \text{km}}}.\end{aligned}$$

At 12 00 hours, a second ship Q leaves a point B with position vector $(19\mathbf{i} + 34\mathbf{j})$ km travelling with velocity vector $(8\mathbf{i} + 6\mathbf{j})$ kmh^{-1}.

(c) Find the velocity of P relative to Q. (2)

Solution

$$(10\mathbf{i} + 10\mathbf{j}) - (8\mathbf{i} + 6\mathbf{j}) = \underline{\underline{(2\mathbf{i} + 4\mathbf{j})}}.$$

(d) Hence, or otherwise, find the time at which P and Q meet and the position vector of the point where this happens. (3)

Solution
Well,

$$\begin{aligned}\text{displacement} &= (19\mathbf{i} + 34\mathbf{j}) - (16\mathbf{i} + 28\mathbf{j}) \\ &= 3\mathbf{i} + 6\mathbf{j}.\end{aligned}$$

Focus on the **i**s:

$$\frac{3}{2} = 1.5$$

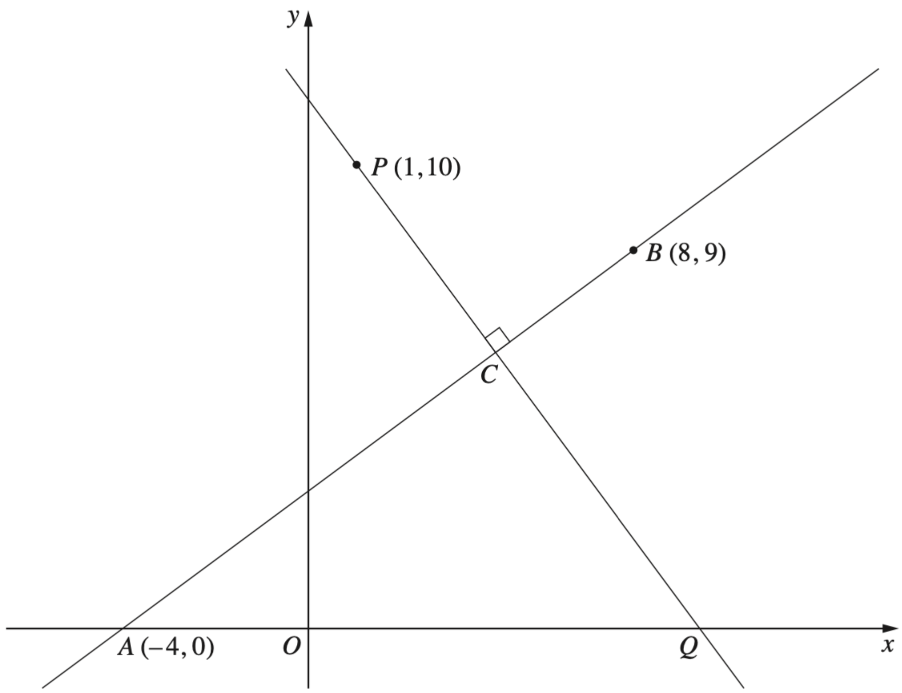
so the time at which P and Q meet is

$$12 + 1.5 = \underline{\underline{13\,30 \text{ hours}}}$$

and they meet at

$$(19\mathbf{i} + 34\mathbf{j}) + 1.5(8\mathbf{i} + 6\mathbf{j}) = \underline{\underline{(31\mathbf{i} + 43\mathbf{j})\text{ km}}}.$$

10. The diagram shows the line AB passing through the points $A(-4, 0)$ and $B(8, 9)$.



The line through the point $P(1, 10)$, perpendicular to AB, meets AB at C and the x-axis at Q.

Find

(a) the coordinates of C and of Q, (7)

Solution

$$m_{AB} = \frac{9-0}{8-(-4)} = \tfrac{3}{4}$$

which means

$$m_{PQ} = -\tfrac{4}{3}.$$

The equation of AB is

$$y - 9 = \tfrac{3}{4}(x - 8) \Rightarrow y - 9 = \tfrac{3}{4}x - 6$$
$$\Rightarrow \boxed{y = \tfrac{3}{4}x + 3 \quad (1).}$$

Now, the equation of PQ is

$$y - 10 = -\tfrac{4}{3}(x - 1) \Rightarrow y - 10 = -\tfrac{4}{3}x + \tfrac{4}{3}$$
$$\Rightarrow \boxed{y = -\tfrac{4}{3}x + \tfrac{34}{3} \quad (2).}$$

Next,

$$\begin{aligned} y = 0 &\Rightarrow -\tfrac{4}{3}x + \tfrac{34}{3} = 0 \\ &\Rightarrow \tfrac{4}{3}x = \tfrac{34}{3} \\ &\Rightarrow x = 8\tfrac{1}{2}; \end{aligned}$$

so, $\underline{\underline{Q(8\tfrac{1}{2}, 0)}}$.

Finally, we want to find C so do (1) = (2):

$$\begin{aligned} \tfrac{3}{4}x + 3 = -\tfrac{4}{3}x + \tfrac{34}{3} &\Rightarrow \tfrac{25}{12}x = \tfrac{25}{3} \\ &\Rightarrow x = 4 \\ &\Rightarrow y = 6; \end{aligned}$$

hence, $\underline{\underline{C(4, 6)}}$.

(b) the area of triangle ACQ. (2)

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2} \times [8\tfrac{1}{2} - (-4)] \times 6 \\ &= \tfrac{1}{2} \times 12\tfrac{1}{2} \times 6 \\ &= \underline{\underline{37\tfrac{1}{2}}}. \end{aligned}$$

11. The table shows experimental values of variables s and t.

t	5	15	30	70	100
s	1 305	349	152	55	36

(a) By plotting a suitable straight line graph, show that s and t are related by the equation (4)

$$s = kt^n,$$

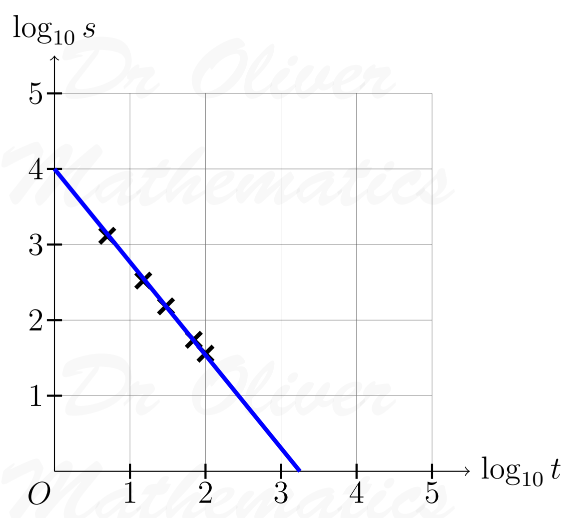
where k and n are constants.

Solution

$$\begin{aligned} s = kt^n &\Rightarrow \log_{10} s = \log_{10}(kt^n) \\ &\Rightarrow \log_{10} s = \log_{10} k + \log_{10}(t^n) \\ &\Rightarrow \log_{10} s = \log_{10} k + n \log_{10} t \end{aligned}$$

so we need

$\log_{10} t$	0.699	1.176	1.477	1.845	2
$\log_{10} s$	3.116	2.523	2.182	1.740	1.556



So the line

$$\log_{10} s = \log_{10} k + n \log_{10} t$$

acts as a line of best fit and so s and t are related by the equation $\underline{\underline{s = kt^n}}$, as required.

(b) Use your graph to find the value of k and of n. (4)

Solution

The line of best fit goes through $(3.2, 0)$ and $(0, 4)$. Well,

$$\begin{aligned} m &= \frac{4-0}{0-3.2} \\ &= \tfrac{5}{4} \end{aligned}$$

and the equation is

$$\begin{aligned} \log_{10} s - 4 = \tfrac{5}{4}(\log_{10} t - 0) &\Rightarrow \log_{10} s - 4 = \log_{10} t^{\frac{5}{4}} \\ &\Rightarrow \log_{10} s - \log_{10} t^{\frac{5}{4}} = 4 \\ &\Rightarrow \log_{10}\left(\frac{s}{t^{\frac{5}{4}}}\right) = 4 \\ &\Rightarrow \frac{s}{t^{\frac{5}{4}}} = 10^4 \\ &\Rightarrow \frac{s}{t^{\frac{5}{4}}} = 10\,000 \\ &\Rightarrow \underline{\underline{s = 10\,000 t^{-\frac{5}{4}}}}; \end{aligned}$$

hence, $\underline{\underline{k = 10\,000}}$ and $\underline{\underline{n = -\tfrac{5}{4}}}$.

(c) Estimate the value of s when $t = 50$. (2)

Solution

$$\begin{aligned} t = 50 \Rightarrow s &= 10\,000\left(50^{-\frac{5}{4}}\right) \\ \Rightarrow s &= 75.212\,061\,86 \text{ (FCD)} \\ \Rightarrow \underline{\underline{s}} &\underline{\underline{= 75.2 \ \text{ (1 dp)}}}. \end{aligned}$$

EITHER

12. (a) State the amplitude of (1)

$$1 + \sin \tfrac{1}{3}x.$$

Solution

$\underline{\underline{1}}$.

(b) State, in radians, the period of (1)

$$1 + \sin \tfrac{1}{3}x.$$

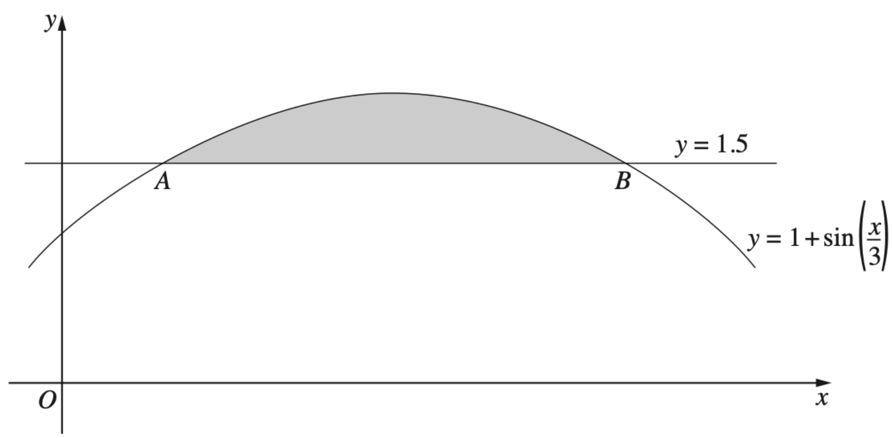
Solution

$$2\pi \times 3 = \underline{\underline{6\pi}}.$$

The diagram shows the curve

$$y = 1 + \sin \tfrac{1}{3}x,$$

meeting the line $y = 1.5$ at points A and B.



Find

(c) the x-coordinate of A and of B, (3)

Solution

$$\begin{aligned} 1 + \sin \tfrac{1}{3}x = 1.5 &\Rightarrow \sin \tfrac{1}{3}x = 0.5 \\ &\Rightarrow \tfrac{1}{3}x = \tfrac{1}{6}\pi\,,\, \tfrac{5}{6}\pi \\ &\Rightarrow x = \tfrac{1}{2}\pi\,,\, \tfrac{5}{2}\pi; \end{aligned}$$

hence, the x-coordinate of A is $\underline{\underline{\tfrac{1}{2}\pi}}$ and the x-coordinate of B is $\underline{\underline{\tfrac{5}{2}\pi}}$.

(d) the area of the shaded region. (6)

Solution

$$\begin{aligned}\text{Area} &= \text{area under the curve} - \text{rectangular area}\\ &= \int_{\frac{1}{2}\pi}^{\frac{5}{2}\pi} (1 + \sin \tfrac{1}{3}x)\,\mathrm{d}x - (1.5 \times 2\pi)\\ &= \left[x - 3\cos \tfrac{1}{3}x\right]_{x=\frac{1}{2}\pi}^{\frac{5}{2}\pi} - 3\pi\\ &= \left(\tfrac{5}{2}\pi + \tfrac{3\sqrt{3}}{2}\right) - \left(\tfrac{1}{2}\pi - \tfrac{3\sqrt{3}}{2}\right) - 3\pi\\ &= \underline{\underline{3\sqrt{3} - \pi}}.\end{aligned}$$

OR

13. A particle moves in a straight line such that t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by

$$v = k\cos 4t,$$

where k is a positive constant.

Find

(a) the value of t when the particle is first instantaneously at rest, (1)

Solution

$$\begin{aligned}v = 0 &\Rightarrow k\cos 4t = 0\\ &\Rightarrow 4t = \tfrac{1}{2}\pi\\ &\Rightarrow \underline{\underline{t = \tfrac{1}{8}\pi}}.\end{aligned}$$

(b) an expression for the acceleration of the particle t s after passing through O. (2)

Solution

$$v = k\cos 4t \Rightarrow \underline{\underline{a = -4k\sin 4t \text{ ms}^{-2}}}.$$

Given that the acceleration of the particle is 12 ms^{-2} when $t = \frac{3}{8}\pi$,

(c) find the value of k. (2)

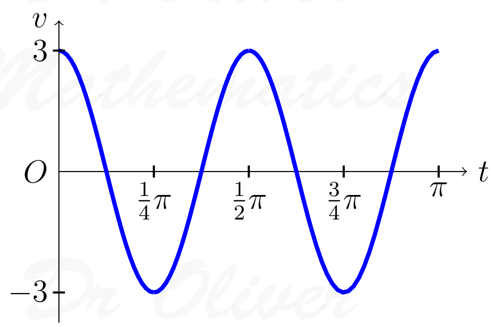
Solution

$$t = \tfrac{3}{8}\pi, a = 12 \Rightarrow -4k\sin[4(\tfrac{3}{8}\pi)] = 12$$
$$\Rightarrow k = -\frac{3}{\sin(\tfrac{3}{2}\pi)}$$
$$\Rightarrow \underline{\underline{k = 3}}.$$

Using your value for k,

(d) sketch the velocity-time curve for the particle for $0 \leqslant t \leqslant \pi$, (2)

Solution



(e) find the displacement of the particle from O when $t = \frac{1}{24}\pi$. (4)

Solution

Well,

$$\int_0^{\frac{1}{24}\pi} (3\cos 4t)\,\mathrm{d}t = \left[\tfrac{3}{4}\sin 4t\right]_{t=0}^{\frac{1}{24}\pi}$$
$$= \tfrac{3}{8} - 0$$
$$= \underline{\underline{\tfrac{3}{8}\text{ m}}}.$$