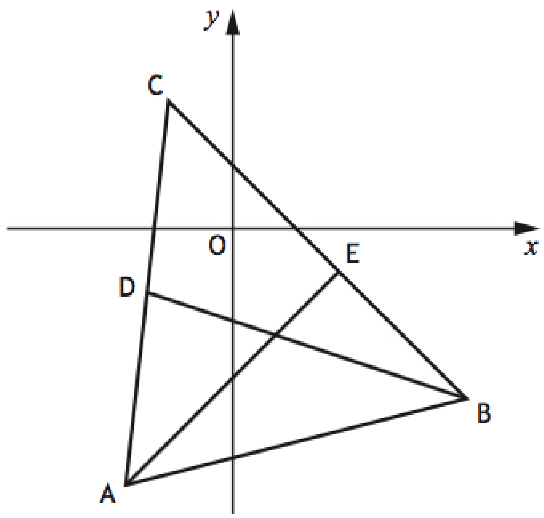
Dr Oliver Mathematics
Mathematics: Higher
2019 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 80.
You must write down all the stages in your working.

1. Triangle ABC has vertices $A(-5,-12)$, $B(11,-8)$, and $C(-3,6)$.



(a) Find the equation of the median BD. (3)

Solution

The coordinates of D are

$$\left(\frac{-5+(-3)}{2}, \frac{-12+6}{2}\right) = (-4,-3).$$

Now, the gradient of BD are

$$\frac{-8-(-3)}{11-(-4)} = \frac{-5}{15} = -\tfrac{1}{3}.$$

Finally, the equation of the median BD is

$$y+8 = -\tfrac{1}{3}(x-11) \Rightarrow y+8 = -\tfrac{1}{3}x + \tfrac{11}{3} \Rightarrow \underline{\underline{y = -\tfrac{1}{3}x - \tfrac{13}{3}}}. \quad (1)$$

(b) Find the equation of the altitude AE. (3)

Solution

$$\text{Gradient of } BC = \frac{6-(-8)}{-3-11} = \frac{14}{-14} = -1$$

and

$$m_{AC} = -\frac{1}{-1} = 1.$$

Finally, the equation of the altitude AE is

$$y + 12 = x + 5 \Rightarrow \underline{\underline{y = x - 7}}. \quad (2)$$

(c) Find the coordinates of the point of intersection of BD and AE. (2)

Solution

Do (2) − (1):

$$0 = \tfrac{4}{3}x - \tfrac{8}{3} \Rightarrow \tfrac{4}{3}x = \tfrac{8}{3} \Rightarrow x = 2 \Rightarrow y = -5;$$

hence, they cross at $\underline{\underline{(2, -5)}}$.

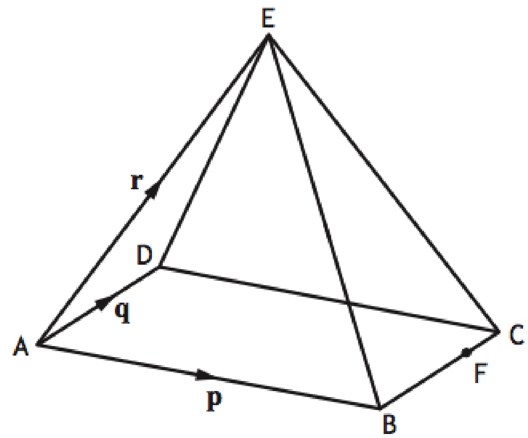
2. Find (4)

$$\int (6\sqrt{x} - 4x^{-3} + 5)\,\mathrm{d}x.$$

Solution

$$\int (6\sqrt{x} - 4x^{-3} + 5)\,\mathrm{d}x = \int (6x^{\frac{1}{2}} - 4x^{-3} + 5)\,\mathrm{d}x = \underline{\underline{4x^{\frac{3}{2}} + 2x^{-2} + 5x + c}}.$$

3. $ABCDE$ is a rectangular based pyramid.
$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$, and $\overrightarrow{AE} = \mathbf{r}$.



(a) Express $\overrightarrow{BE}$ in terms of $\mathbf{p}$ and $\mathbf{r}$. (1)

Solution

$$\begin{aligned}\overrightarrow{BE} &= \overrightarrow{BA} + \overrightarrow{AE} \\ &= \underline{\underline{-\mathbf{p} + \mathbf{r}}}.\end{aligned}$$

Point F divides BC in the ratio $3:1$.

(b) Express vector $\overrightarrow{EF}$ in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. (2)

Solution

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF} \\ &= \overrightarrow{EA} + \overrightarrow{AB} + \tfrac{3}{4}\overrightarrow{BC} \\ &= \underline{\underline{-\mathbf{r} + \mathbf{p} + \tfrac{3}{4}\mathbf{q}}}.\end{aligned}$$

4. In a forest, the population of a species of mouse is falling by 2.7% each year.

To increase the population scientists plan to release 30 mice into the forest at the end of March each year.

u_n is the estimated population of mice at the start of April, n years after the population was first estimated.

It is known that u_n and u_{n+1} satisfy the recurrence relation

$$u_{n+1} = au_n + b.$$

(a) State the values of a and b. (1)

Solution
$a = 1 - 0.027 = \underline{\underline{0.973}}$ and $\underline{\underline{b = 30}}$.

The scientists continue to release this species of mouse each year.

(b) (i) Explain why the estimated population of mice will stabilise in the long term. (1)

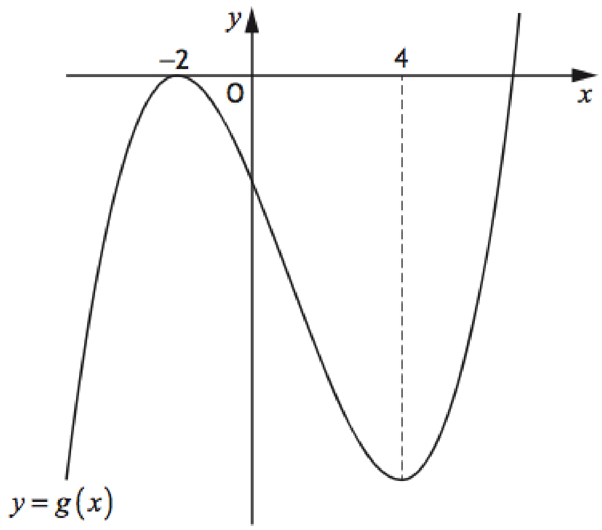
Solution
A limit exists as the recurrence relation is $\underline{\underline{\text{linear}}}$ and $|0.973| < 1$.

(ii) Calculate the long term population to the nearest hundred. (2)

Solution
Let the long term be u. Then

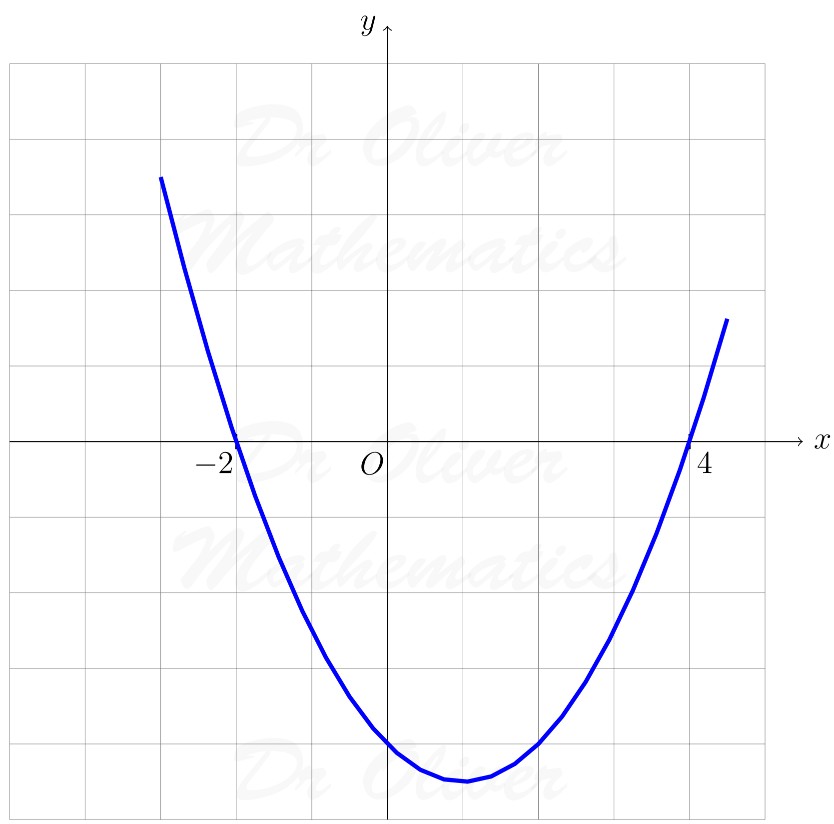
$$\begin{aligned} u = 0.973u + 30 &\Rightarrow 0.027u = 30 \\ &\Rightarrow u = 1\,111\tfrac{1}{9} \\ &\Rightarrow \underline{\underline{u = 1\,100 \ \text{(nearest hundred)}}}. \end{aligned}$$

5. The diagram below shows the graph of a cubic function $y = \text{g}(x)$, with stationary points at $x = -2$ and $x = 4$. (2)



Sketch the graph of $y = \mathrm{g}'(x)$.

Solution



6. (a) Express (4)

$$2\cos x^\circ - 3\sin x^\circ$$

in the form

$$k\cos(x+a)^\circ$$

where $k > 0$ and $0 \leqslant a < 360$.

Solution

$$\begin{aligned} 2\cos x^\circ - 3\sin x^\circ &\equiv k\cos(x+a)^\circ \\ &\equiv k\cos x^\circ \cos a^\circ - k\sin x^\circ \sin a^\circ \end{aligned}$$

and

$$k\cos a^\circ = 2 \text{ and } k\sin a^\circ = 3.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{k^2(\cos^2 a^\circ + \sin^2 a^\circ)} \\ &= \sqrt{(k\cos a^\circ)^2 + (k\sin a^\circ)^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

and

$$\begin{aligned} \tan a^\circ = \frac{k\sin a^\circ}{k\cos a^\circ} &\Rightarrow \tan a^\circ = \tfrac{3}{2} \\ &\Rightarrow a = 56.309\,932\,47 \text{ (FCD)}; \end{aligned}$$

hence,

$$2\cos x^\circ - 3\sin x^\circ = \underline{\underline{\sqrt{13}\cos(x + 56.309\ldots)^\circ}}.$$

(b) Hence solve (3)

$$2\cos x^\circ - 3\sin x^\circ = 3$$

for $0 \leqslant x < 360$.

Solution

$$\begin{aligned} 2\cos x^\circ - 3\sin x^\circ = 3 &\Rightarrow \sqrt{13}\cos(x + 56.309\ldots)^\circ = 3 \\ &\Rightarrow \cos(x + 56.309\ldots)^\circ = \frac{3}{\sqrt{13}} \\ &\Rightarrow x + 56.309\ldots = 326.309\,932\,5,\ 396.690\,067\,53 \text{ (FCD)} \\ &\Rightarrow x = 270 \text{ (exact!)},\ 337.380\,135\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 270,\ 337 \text{ (3 sf)}}}. \end{aligned}$$

7. (a) Express (3)

$$-6x^2 + 24x - 25$$

in the form

$$p(x+q)^2 + r.$$

Solution

$$\begin{aligned} -6x^2 + 24x - 25 &\equiv -6[x^2 - 4x] - 25 \\ &\equiv -6[(x^2 - 4x + 4) - 4] - 25 \\ &\equiv -6[(x-2)^2 - 4] - 25 \\ &\equiv -6(x-2)^2 + 24 - 25 \\ &\equiv \underline{\underline{-6(x-2)^2 - 1}}; \end{aligned}$$

hence, $\underline{\underline{p = -6}}$, $\underline{\underline{q = -2}}$, and $\underline{\underline{r = -1}}$.

(b) Given that (3)

$$\mathrm{f}(x) = -2x^3 + 12x^2 - 25x + 9,$$

show that $\mathrm{f}(x)$ is strictly decreasing for all $x \in \mathbb{R}$.

Solution

$$\begin{aligned} \mathrm{f}(x) = -2x^3 + 12x^2 - 25x + 9 &\Rightarrow \mathrm{f}'(x) = -6x^2 + 24x - 25 \\ &\Rightarrow \mathrm{f}'(x) = -6(x-2)^2 - 1 \\ &\Rightarrow \mathrm{f}'(x) \leqslant -1 \text{ for all } x \in \mathbb{R} \end{aligned}$$

and so $\mathrm{f}(x)$ is <u>strictly decreasing</u> for all $x \in \mathbb{R}$.

8. A function, f, is given by

$$f(x) = \sqrt[3]{x} + 8.$$

The domain of f is $1 \leqslant x \leqslant 1\,000$, $x \in \mathbb{R}$.
The inverse function, f^{-1}, exists.

(a) Find $f^{-1}(x)$. (3)

Solution

$$y = \sqrt[3]{x} + 8 \Rightarrow \sqrt[3]{x} = y - 8$$
$$\Rightarrow x = (y-8)^3;$$

hence,

$$f^{-1}(x) = \underline{\underline{(x-8)^3}}.$$

(b) State the domain of f^{-1}. (1)

Solution
Now,

$$f(1) = \sqrt[3]{1} + 8 = 9 \text{ and } f(1\,000) = \sqrt[3]{1\,000} + 8 = 18$$

which means

$$\underline{\underline{9 \leqslant x \leqslant 18}}.$$

9. Electricity on a spacecraft can be produced by a type of nuclear generator.
The electrical power produced by this generator can be modelled by

$$P_t = 120e^{-0.007\,9t},$$

where P_t is the electrical power produced, in watts, after t years.

(a) Determine the electrical power initially produced by the generator. (1)

Solution
$P_0 = \underline{\underline{120 \text{ watts}}}$.

(b) Calculate how long it takes for the electrical power produced by the generator to reduce by 15%. (4)

Solution

$$\frac{85}{100} \times 120 = 102 \text{ watts}$$

and

$$\begin{aligned}102 = 120\mathrm{e}^{-0.0079t} &\Rightarrow \mathrm{e}^{-0.0079t} = \tfrac{85}{100} \\ &\Rightarrow -0.0079t = \ln \tfrac{17}{20} \\ &\Rightarrow t = \frac{\ln \frac{17}{20}}{-0.0079} \\ &\Rightarrow t = 20.57201639 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 20.6 \text{ years (3 sf)}}}.\end{aligned}$$

10. (a) Show that $(x+3)$ is a factor of (2)

$$3x^4 + 10x^3 + x^2 - 8x - 6.$$

Solution

-3	3	10	1	-8	-6
	$\downarrow$	-9	-3	6	6
	3	1	-2	-2	0

Hence, because there is no remainder, $(x+3)$ is a factor of $3x^4+10x^3+x^2-8x-6$.

(b) Hence, or otherwise, factorise (5)

$$3x^4 + 10x^3 + x^2 - 8x - 6$$

fully.

Solution

Well,

$$3x^4 + 10x^3 + x^2 - 8x - 6 = (x+3)(3x^3 + x^2 - 2x - 2)$$

and it remains to find a linear factor for

$$\mathrm{g}(x) = 3x^3 + x^2 - 2x - 2.$$

$\underline{x = 1}$: $g(1) = 3 + 1 - 2 - 2 = 0$ and we have found a factor!

1	3	1	−2	−2
	↓	3	4	2
	3	4	2	0

So,

$$3x^4 + 10x^3 + x^2 - 8x - 6 = (x + 3)(x - 1)(3x^2 + 4x + 2).$$

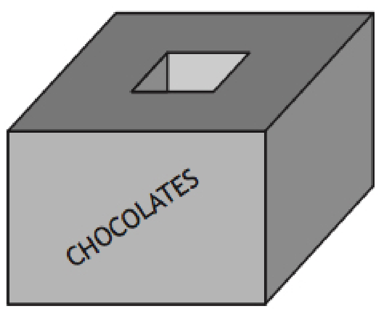
Now, $a = 3$, $b = 4$, and $c = 2$:

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4 \times 3 \times 2 \\ &= -8 \\ &< 0 \end{aligned}$$

and the quadratic have no real roots. Hence,

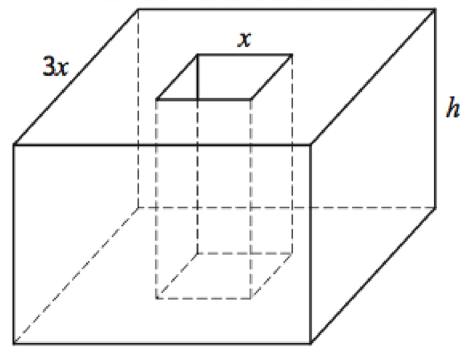
$$3x^4 + 10x^3 + x^2 - 8x - 6 = \underline{\underline{(x + 3)(x - 1)(3x^2 + 4x + 2)}}.$$

11. A manufacturer of chocolates is launching a new product in novelty shaped cardboard boxes.



The box is a cuboid with a cuboid shaped tunnel through it.

- The height of the box is h centimetres
- The top of the box is a square of side $3x$ centimetres
- The end of the tunnel is a square of side x centimetres.
- The volume of the box is $2\,000$ cm^3.



(a) Show that the total surface area, A cm^2, of the box is given by (3)

$$A = 16x^2 + \frac{4\,000}{x}.$$

Solution

$$\begin{aligned}\text{Cross-sectional area} &= (3x)^2 - x^2 \\ &= 9x^2 - x^2 \\ &= 8x^2\end{aligned}$$

and

$$\begin{aligned}\text{volume} = \text{cross-sectional area} \times \text{height} &\Rightarrow 2\,000 = 8x^2\,h \\ &\Rightarrow h = \frac{2\,000}{8x^2}.\end{aligned}$$

Now,

$$\begin{aligned}A &= (4 \cdot \text{sides}) + (2 \cdot \text{top and bottom}) + (4 \cdot \text{hollow portion}) \\ &= (4 \cdot (3x)h) + (2 \cdot 8x^2) + (4 \cdot xh) \\ &= 12xh + 16x^2 + 4xh \\ &= 16x^2 + 16xh \\ &= 16x^2 + 16x\left(\frac{2\,000}{8x^2}\right) \\ &= \underline{\underline{16x^2 + \frac{4\,000}{x}}},\end{aligned}$$

as required.

To minimise the cost of production, the surface area, A, of the box should be as small as possible.

(b) Find the minimum value of A. (6)

Solution

$$A = 16x^2 + \frac{4\,000}{x} \Rightarrow A = 16x^2 + 4\,000x^{-1}$$
$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = 32x - 4\,000x^{-2}$$
$$\Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 32 + 8\,000x^{-3}.$$

and

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 32x - 4\,000x^{-2} = 0$$
$$\Rightarrow 32x = 4\,000x^{-2}$$
$$\Rightarrow x^3 = 125$$
$$\Rightarrow x = 5.$$

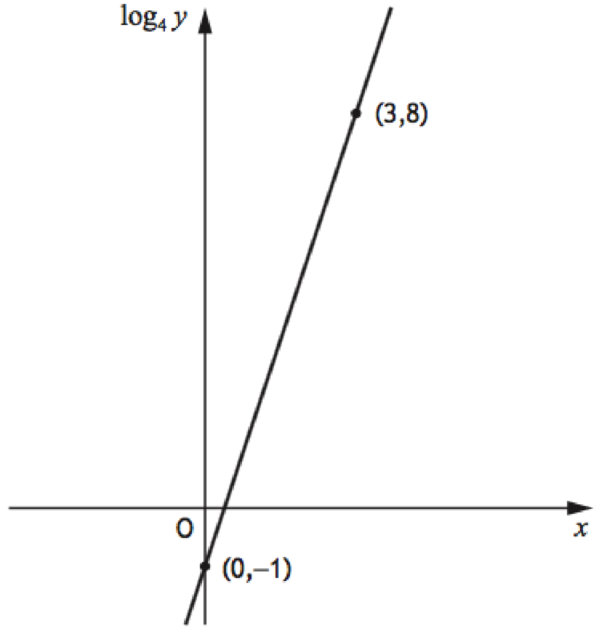
Now,

$$x = 5 \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 96 > 0$$

and hence $x = 5$ is a minimum value. Hence,

$$A = 16(5^2) + \frac{4\,000}{5}$$
$$= 400 + 800$$
$$= \underline{\underline{1\,200 \text{ cm}^2}}.$$

12. Two variables, x and y, are connected by the equation $y = ab^x$. (5)
The graph of $\log_4 y$ against x is a straight line as shown.



Find the values of a and b.

Solution

$$\begin{aligned}\text{Gradient} &= \frac{8-(-1)}{3-0}\\ &= \frac{9}{3}\\ &= 3\end{aligned}$$

and the equation is

$$\begin{aligned}\log_4 y + 1 = 3(x-0) &\Rightarrow \log_4 y = 3x - 1\\ &\Rightarrow y = 4^{3x-1}\\ &\Rightarrow y = (4^3)^x \cdot 4^{-1}\\ &\Rightarrow \underline{\underline{y = \tfrac{1}{4}\cdot 64^x}};\end{aligned}$$

hence, $\underline{\underline{a = \frac{1}{4}}}$ and $\underline{\underline{b = 64}}$.

13. For a function, f, defined on the set of real numbers, $\mathbb{R}$, it is known that (5)

- the rate of change of f with respect to x is given by

$$3x^2 - 16x + 11.$$

- the graph with equation $y = \mathrm{f}(x)$ crosses the x-axis at $(7, 0)$.

Express $\mathrm{f}(x)$ in terms of x.

Solution

$$\mathrm{f}'(x) = 3x^2 - 16x + 11 \Rightarrow \mathrm{f}(x) = x^3 - 8x^2 + 11x + c$$

for some constant c. Now,

$$\begin{aligned} x = 7, y = 0 &\Rightarrow 0 = 343 - 392 + 77 + c \\ &\Rightarrow c = -28 \end{aligned}$$

and so

$$\underline{\underline{\mathrm{f}(x) = x^3 - 8x^2 + 11x - 28}}.$$

14. The vectors **u** and **v** are such that (4)

- $|\mathbf{u}| = 4$.
- $|\mathbf{v}| = 5$
- $\mathbf{u}.(\mathbf{u} + \mathbf{v}) = 21$.

Determine the size of the angle between the vectors **u** and **v**.

Solution

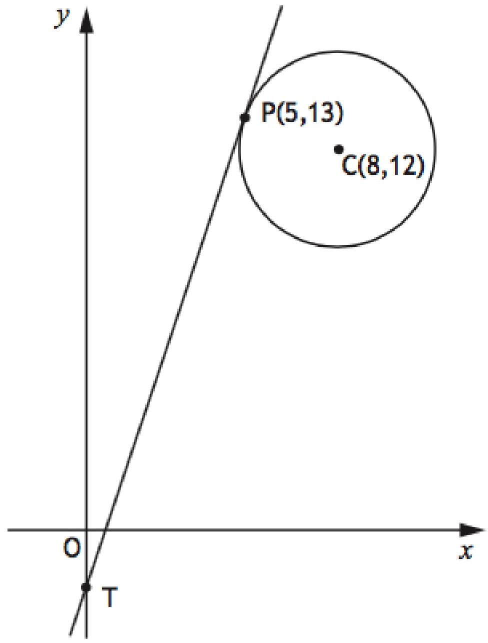
Let θ° be the angle between the vectors **u** and **v**.

$$\begin{aligned} \mathbf{u}.(\mathbf{u} + \mathbf{v}) = 21 &\Rightarrow \mathbf{u}.\mathbf{u} + \mathbf{u}.\mathbf{v} = 21 \\ &\Rightarrow 16 + \mathbf{u}.\mathbf{v} = 21 \\ &\Rightarrow \mathbf{u}.\mathbf{v} = 5. \end{aligned}$$

Now,

$$\begin{aligned} & \mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta^\circ \\ \Rightarrow\ & 5 = 4 \cdot 5 \cdot \cos \theta^\circ \\ \Rightarrow\ & \cos \theta^\circ = \tfrac{1}{4} \\ \Rightarrow\ & \theta = 75.522\,487\,81 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\theta = 75.5 \text{ (3 sf)}}}. \end{aligned}$$

15. A circle has centre $C(8, 12)$.
The point $P(5, 13)$ lies on the circle as shown.



(a) Find the equation of the tangent at P. (3)

Solution

$$m_{CP} = \frac{13-12}{5-8} = -\frac{1}{3}$$

which means the gradient is the perpendicular is

$$-\frac{1}{-\frac{1}{3}} = 3.$$

Finally, the equation of the tangent is

$$y - 13 = 3(x-5) \Rightarrow y - 13 = 3x - 15 \Rightarrow \underline{\underline{y = 3x - 2}}.$$

The tangent from P meets the y-axis at the point T.

(b) (i) State the coordinates of T. (1)

Solution
$\underline{\underline{T(0, -2)}}$.

(ii) Find the equation of the circle that passes through the points C, P, and T. (3)

Solution
As $\triangle CPT$ is a right-angled triangle with the 90° at P, the circle go will through M, the midpoint CT:

$$\left(\frac{8+0}{2}, \frac{12+(-2)}{2}\right) = (4, 5).$$

Now,

$$\begin{aligned} MT^2 &= (4-0)^2 + [5-(-2)]^2 \\ &= 65 \end{aligned}$$

and, finally, the equation is

$$\underline{\underline{(x-4)^2 + (y-7)^2 = 65}}.$$