Dr Oliver Mathematics
Further Mathematics: Further Pure Mathematics 1 (Paper 3A)
November 2021: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. The ellipse E has equation
$$\frac{x^2}{36} + \frac{y^2}{20} = 1.$$
Find
   (a) the coordinates of the foci of E, (3)
   (b) the equations of the directrices of E. (2)

2. (a) Use the substitution (5)
$$t = \tan \tfrac{1}{2}x$$
to prove the identity
$$\frac{\sin x - \cos x + 1}{\sin x + \cos x + 1} \equiv \sec x + \tan x,\ x \neq \tfrac{1}{2}n\pi,\ n \in \mathbb{Z}.$$
   (b) Use the substitution (5)
$$t = \tan \tfrac{1}{2}\theta$$
to determine the exact value of
$$\int_0^{\frac{1}{2}\pi} \left(\frac{5}{4 + 2\cos\theta} \right) \mathrm{d}\theta,$$
giving your answer in simplest form.

3. Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where (8)
$$\mathrm{f}(x) = \frac{x}{|x| - 2}.$$

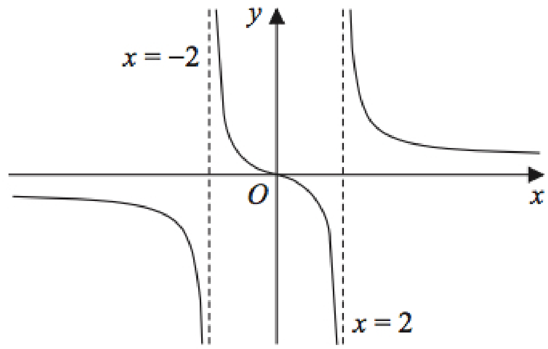

Figure 1: $\mathrm{f}(x) = \dfrac{x}{|x| - 2}$

Use algebra to determine the values of x for which

$$2x - 5 > \frac{x}{|x| - 2}.$$

4. A small aircraft is landing in a field.

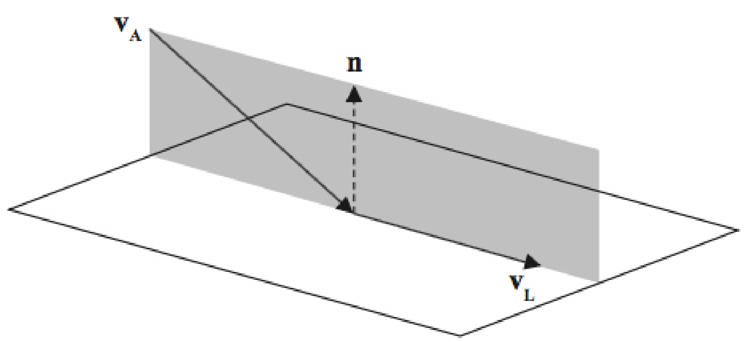

Figure 2: a small aircraft

In a model for the landing the aircraft travels in different straight lines before and after it lands, as shown in Figure 2.

The vector $\mathbf{v}_A$ is in the direction of travel of the aircraft as it approaches the field.

The vector $\mathbf{v}_L$ is in the direction of travel of the aircraft after it lands.

With respect to a fixed origin, the field is modelled as the plane with equation

$$x - 2y + 25z = 0$$

and

$$\mathbf{v}_A = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

(a) Write down a vector $\mathbf{n}$ that is a normal vector to the field. (1)

(b) Show that (2)

$$\mathbf{n} \times \mathbf{v}_A = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix},$$

where λ is a constant to be determined.

When the aircraft lands it remains in contact with the field and travels in the direction $\mathbf{v}_L$.

The vector $\mathbf{v}_L$ is in the same plane as both $\mathbf{v}_A$ and $\mathbf{n}$ as shown in Figure 2.

(c) Determine a vector which has the same direction as $\mathbf{v}_L$. (3)

(d) State a limitation of the model. (1)

5. The parabola C has equation

$$y^2 = 32x$$

and the hyperbola H has equation

$$\frac{x^2}{36} - \frac{y^2}{9} = 1.$$

(a) Write down the equations of the asymptotes of H. (1)

The line l_1 is normal to C and parallel to the asymptote of H with positive gradient.

The line l_2 is normal to C and parallel to the asymptote of H with negative gradient.

(b) Determine (4)

(i) an equation for l_1,

(ii) an equation for l_2.

The lines l_1 and l_2 meet H at the points P and Q respectively.

(c) Find the area of the triangle OPQ, where O is the origin. (4)

6. Given that

$$y = (1 + \ln x)^2,\ x > 0,$$

(a) show that (4)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{2\ln x}{x^2}.$$

(b) Hence find (2)

$$\frac{d^3y}{dx^3}.$$

(c) Determine the Taylor series expansion about $x = 1$ of (3)

$$(1 + \ln x)^2$$

in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$.

Give each coefficient in simplest form.

(d) Use this series expansion to evaluate (3)

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x - 1)^3},$$

explaining your reasoning clearly.

7. With respect to a fixed origin O, the line l has equation (7)

$$(\mathbf{r} - (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k})) \times (9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = \mathbf{0}.$$

The point A lies on l such that the direction cosines of OA with respect to the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ axes are $\frac{3}{7}$, β, and γ.

Determine the coordinates of the point A.

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \tfrac{1}{3} x \tanh t \quad \text{(I)}.$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula (4)

$$\left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_n)}{h},$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation (3)

$$\frac{du}{dt} + u \tanh t = 1 + \frac{3}{\cosh t} \quad \text{(II)}.$$

(c) Determine the general solution of equation (II). (4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied. (3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate. (3)