Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2005 November Paper 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Variables V and t are related by the equation

$$V = 1\,000\mathrm{e}^{-kt},$$

where k is a constant.

Given that $V = 500$ when $t = 21$, find

(a) the value of k, (2)

Solution

$$\begin{aligned} V = 500, t = 21 &\Rightarrow 500 = 1\,000\mathrm{e}^{-21k} \\ &\Rightarrow \mathrm{e}^{-21k} = \tfrac{1}{2} \\ &\Rightarrow -21k = \ln \tfrac{1}{2} \\ &\Rightarrow k = -\tfrac{1}{21} \ln \tfrac{1}{2} \\ &\Rightarrow k = 0.033\,007\,008\,6 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{k = 0.033\,0 \text{ (3 sf)}}}. \end{aligned}$$

(b) the value of V when $t = 30$. (2)

Solution

Now,

$$\begin{aligned} V &= 1\,000\mathrm{e}^{-0.033\ldots \times 30} \\ &= 371.498\,572\,3 \text{ (FCD)} \\ &= \underline{\underline{371 \text{ (nearest whole number)}}}. \end{aligned}$$

2. The line (5)

$$x + y = 10$$

meets the curve

$$y^2 = 2x + 4$$

at the points A and B.

Find the coordinates of the mid-point of AB.

Solution

Now,

$$x + y = 10 \Rightarrow y = 10 - x$$

and

$$y^2 = 2x + 4 \Rightarrow (10 - x)^2 = 2x + 4$$

$\times$	10	$-x$
10	100	$-10x$
$-x$	$-10x$	$+x^2$

e.g.,

$$\Rightarrow 100 - 20x + x^2 = 2x + 4$$
$$\Rightarrow x^2 - 22x + 96 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -22 \\ \text{multiply to:} & +96\end{array}\right\} -6,\ -16$$

$$\Rightarrow (x - 6)(x - 16) = 0$$
$$\Rightarrow x = 6 \text{ or } x = 16$$
$$\Rightarrow y = 4 \text{ or } y = -6;$$

so, $A(6, 4)$ and $B(16, -6)$.

Finally,

$$AB = \sqrt{(16 - 6)^2 + (-6 - 4)^2}$$
$$= \sqrt{200}$$
$$= \underline{\underline{2\sqrt{10}}}.$$

3. (a) Given that (2)

$$y = 1 + \ln(2x - 3),$$

obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = 1 + \ln(2x-3) \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x-3}}}.$$

(b) Hence find, in terms of p, the approximate value of y when $x = 2 + p$, where p is small. (3)

Solution

Well,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and

$$\begin{aligned}\delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= \underline{\underline{2p}}.\end{aligned}$$

4. The function f is given by

$$\mathrm{f} : x \mapsto 2 + 5\sin 3x, \text{ for } 0^\circ \leqslant x \leqslant 180^\circ.$$

(a) State the amplitude and period of f. (2)

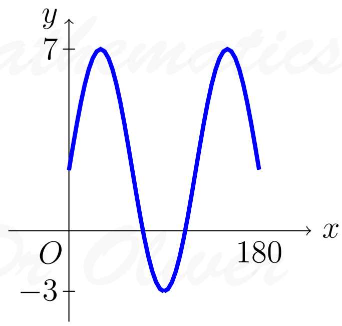
Solution

The amplitude is $\underline{\underline{5}}$ and the period is

$$\frac{360}{3} = \underline{\underline{120^\circ}}.$$

(b) Sketch the graph of $y = \mathrm{f}(x)$. (3)

Solution



5. The binomial expansion of (6)

$$(1 + px)^n,$$

where $n > 0$, in ascending powers of x is

$$1 - 12x + 28p^2x^2 + qx^3 + \dots.$$

Find the value of n, of p, and of q.

Solution

$$\begin{aligned}(1 + px)^n &= 1 + (n)(px) + \frac{n(n-1)}{2!}(px)^2 + \frac{n(n-1)(n-2)}{3!}(px)^3 + \dots \\ &= 1 + npx + \tfrac{1}{2}n(n-1)p^2x^2 + \tfrac{1}{6}n(n-1)(n-2)p^3x^3 + \dots\end{aligned}$$

x^2 term:

$$\begin{aligned}\tfrac{1}{2}n(n-1) = 28 &\Rightarrow n(n-1) = 56 \\ &\Rightarrow n^2 - n = 56 \\ &\Rightarrow n^2 - n - 56 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -56\end{array}\right\} -8,\ +7$$

e.g.,

$$\begin{aligned}&\Rightarrow (n-8)(n+7) = 0 \\ &\Rightarrow n = 8 \text{ or } n = -7.\end{aligned}$$

But $n > 0$ so $\underline{\underline{n = 8}}$.

Now, x term:

$$np = -12 \Rightarrow p = -\frac{12}{8}$$
$$\Rightarrow \underline{\underline{p = -\tfrac{3}{2}}}.$$

Finally, x^3 term:

$$\begin{aligned} q &= \tfrac{1}{6}n(n-1)(n-2)p^3 \\ &= \tfrac{1}{6}(8)(7)(6)(-\tfrac{3}{2})^3 \\ &= \underline{\underline{-189}}. \end{aligned}$$

6. It is given that (6)

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & p \end{pmatrix}$$

and that

$$\mathbf{A} + \mathbf{A}^{-1} = k\mathbf{I},$$

where p and k are constants and $\mathbf{I}$ is the identity matrix.

Evaluate p and k.

Solution

Well,

$$\det \mathbf{A} = 3k - 5$$

and

$$\mathbf{A}^{-1} = \tfrac{1}{3k-5} \begin{pmatrix} p & -1 \\ -5 & 3 \end{pmatrix}.$$

Now,

$$\begin{aligned} \mathbf{A} + \mathbf{A}^{-1} &= \begin{pmatrix} 3 & 1 \\ 5 & p \end{pmatrix} + \tfrac{1}{3k-5} \begin{pmatrix} p & -1 \\ -5 & 3 \end{pmatrix} \\ &= \tfrac{1}{3k-5} \left[(3k-5) \begin{pmatrix} 3 & 1 \\ 5 & p \end{pmatrix} + \begin{pmatrix} p & -1 \\ -5 & 3 \end{pmatrix} \right] \\ &= \tfrac{1}{3k-5} \left[\begin{pmatrix} 9k-15 & 3k-5 \\ 15k-25 & 3kp-5p \end{pmatrix} + \begin{pmatrix} p & -1 \\ -5 & 3 \end{pmatrix} \right] \\ &= \tfrac{1}{3k-5} \begin{pmatrix} 9k-15+p & 3k-6 \\ 15k-30 & 3kp-5p+3 \end{pmatrix} \end{aligned}$$

Now,

$$\begin{aligned}(1,2) \text{ entry} = 0 &\Rightarrow 3k - 6 = 0 \\ &\Rightarrow 3k = 6 \\ &\Rightarrow \underline{\underline{k = 2}}.\end{aligned}$$

Next, $(1,1)$ entry:

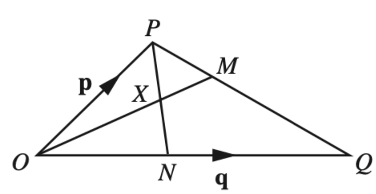
$$\begin{aligned}\tfrac{1}{3k-5}(9k - 15 + p) &= \tfrac{1}{3(2)-5}[9(2) - 15 + p] \\ &= 3 + p\end{aligned}$$

and

$$3 + p = k \Rightarrow \underline{\underline{p = -2}}.$$

7. In the diagram,

- $\overrightarrow{OP} = \mathbf{p}$,
- $\overrightarrow{OQ} = \mathbf{q}$,
- $\overrightarrow{PM} = \frac{1}{3}\overrightarrow{PQ}$, and
- $\overrightarrow{ON} = \frac{2}{5}\overrightarrow{OQ}$.



(a) Given that $\overrightarrow{OX} = m\overrightarrow{OM}$, express $\overrightarrow{OX}$ in terms of m, $\mathbf{p}$, and $\mathbf{q}$. (2)

Solution

$$\begin{aligned}
\overrightarrow{OX} &= m\overrightarrow{OM} \\
&= m(\overrightarrow{OP} + \overrightarrow{PM}) \\
&= m(\overrightarrow{OP} + \tfrac{1}{3}\overrightarrow{PQ}) \\
&= m[\overrightarrow{OP} + \tfrac{1}{3}(\overrightarrow{PO} + \overrightarrow{OQ})] \\
&= m[\overrightarrow{OP} + \tfrac{1}{3}(-\overrightarrow{OP} + \overrightarrow{OQ})] \\
&= m[\mathbf{p} + \tfrac{1}{3}(-\mathbf{p} + \mathbf{q})] \\
&= m(\tfrac{2}{3}\mathbf{p} + \tfrac{1}{3}\mathbf{q}) \\
&= \underline{\underline{\tfrac{1}{3}m(2\mathbf{p} + \mathbf{q})}}.
\end{aligned}$$

(b) Given that $\overrightarrow{PX} = n\overrightarrow{PN}$, express $\overrightarrow{OX}$ in terms of n, $\mathbf{p}$, and $\mathbf{q}$. (3)

Solution

Now,

$$\begin{aligned}
\overrightarrow{PN} &= \overrightarrow{PO} + \overrightarrow{ON} \\
&= -\overrightarrow{OP} + \tfrac{2}{5}\overrightarrow{OQ} \\
&= -\mathbf{p} + \tfrac{2}{5}\mathbf{q}
\end{aligned}$$

so

$$\begin{aligned}
\overrightarrow{OX} &= \overrightarrow{OP} + \overrightarrow{PX} \\
&= \overrightarrow{OP} + n\overrightarrow{PN} \\
&= \mathbf{p} + n(-\mathbf{p} + \tfrac{2}{5}\mathbf{q}) \\
&= \underline{\underline{(1 - n)\mathbf{p} + \tfrac{2}{5}\mathbf{q}}}.
\end{aligned}$$

(c) Hence evaluate m and n. (2)

Solution

We will take $\mathbf{p}$ and $\mathbf{q}$ separately:

$$\begin{aligned}
\tfrac{2}{3}m &= 1 - n \quad (1) \\
\tfrac{1}{3}m &= \tfrac{2}{5} \quad (2).
\end{aligned}$$

Now,

$$\tfrac{1}{3}m = \tfrac{2}{5} \Rightarrow \underline{\underline{m = \tfrac{6}{5}}}$$

and

$$\tfrac{2}{3}(\tfrac{6}{5}) = 1 - n \Rightarrow n = 1 - \tfrac{4}{5}$$
$$\Rightarrow \underline{\underline{n = \tfrac{1}{5}}}.$$

8. (a) Find the value of each of the integers p and q for which (2)

$$\left(\frac{25}{16}\right)^{-\frac{3}{2}} = 2^p \times 5^q.$$

Solution

Well,

$$\begin{aligned}\left(\frac{25}{16}\right)^{-\frac{3}{2}} &= \left(\frac{16}{25}\right)^{\frac{3}{2}} \\ &= \left[\left(\frac{16}{25}\right)^{\frac{1}{2}}\right]^3 \\ &= \left[\left(\frac{\sqrt{16}}{\sqrt{25}}\right)\right]^3 \\ &= \left[\left(\frac{4}{5}\right)\right]^3 \\ &= \frac{4^3}{5^3} \\ &= \frac{(2^2)^3}{5^3} \\ &= \frac{2^6}{5^3} \\ &= \underline{\underline{2^6 \times 5^{-3}}};\end{aligned}$$

hence, $\underline{\underline{p = 6}}$ and $\underline{\underline{q = -3}}$.

(b) (i) Express the equation (2)

$$4^x - 2^{x+1} = 3,$$

as a quadratic equation in 2^x.

Solution

$$\begin{aligned} 4^x - 2^{x+1} = 3 &\Rightarrow (2^2)^x - 2^x \cdot 2^1 - 3 = 0 \\ &\Rightarrow \underline{\underline{(2^x)^2 - 2 \cdot 2^x - 3 = 0}}, \end{aligned}$$

as required.

(ii) Hence find the value of x, correct to 2 decimal places. (3)

Solution
Let $y = 2^x$. Then

$$(2^x)^2 - 2 \cdot 2^x - 3 = 0 \Rightarrow y^2 - 2y - 3 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -3 \end{array}\right\} -3,\ +1$$

e.g.,

$$\begin{aligned} &\Rightarrow (y-3)(y+1) = 0 \\ &\Rightarrow y = 3 \text{ or } y = -1 \\ &\Rightarrow 2^x = 3 \text{ or } 2^x = -1 \text{ (not possible)} \\ &\Rightarrow x = \log_2 3 \\ &\Rightarrow x = 1.584\,962\,501 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 1.58 \text{ (2 dp)}}}. \end{aligned}$$

9. The function

$$\mathrm{f}(x) = x^3 - 6x^2 + ax + b,$$

where a and b are constants, is exactly divisible by $(x-3)$ and leaves a remainder of -55 when divided by $(x+2)$.

(a) Find the value of a and of b. (4)

Solution
We use synthetic division twice:

3	1	-6	a	b
	$\downarrow$	3	-9	$3a-27$
	1	-3	$a-9$	$3a+b-27$

and so

$$3a + b - 27 = 0 \Rightarrow 3a + b = 27 \quad (1)$$

and

-2	1	-6	a	b
	$\downarrow$	-2	16	$-2a - 32$
	1	-8	$a + 16$	$-2a + b - 32$

and so

$$-2a + b - 32 = -55 \Rightarrow -2a + b = -23 \quad (2).$$

Do (1) − (2):

$$\begin{aligned} 5a = 50 &\Rightarrow \underline{\underline{a = 10}} \\ &\Rightarrow 3(10) + b = 27 \\ &\Rightarrow 30 + b = 27 \\ &\Rightarrow \underline{\underline{b = -3}}. \end{aligned}$$

(b) Solve the equation $f(x) = 0$. (4)

Solution
Well,

$$x^3 - 6x^2 + 10x - 3 = 0 \Rightarrow (x - 3)(x^2 - 3x + 1) = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

$$\Rightarrow \underline{\underline{x = 3, \frac{3 \pm \sqrt{5}}{2}}}.$$

10. A curve is such that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x - 2.$$

The gradient of the curve at the point $(2, -9)$ is 3.

(a) Express y in terms of x. (5)

Solution

Well,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x - 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x + c,$$

for some constant c. Now,

$$\begin{aligned} x = 2, \frac{\mathrm{d}y}{\mathrm{d}x} = 3 &\Rightarrow 3 = 3(2^2) - 2(2) + c \\ &\Rightarrow 3 = 12 - 4 + c \\ &\Rightarrow c = -5 \end{aligned}$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x - 5.$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 2x - 5 \Rightarrow y = x^3 - x^2 - 5x + d,$$

for some constant d. Well,

$$\begin{aligned} x = 2, y = -9 &\Rightarrow -9 = 2^3 - 2^2 - 5(2) + d \\ &\Rightarrow -9 = 8 - 4 - 10 + d \\ &\Rightarrow d = -3 \end{aligned}$$

so

$$\underline{\underline{y = x^3 - x^2 - 5x - 3}}.$$

(b) Show that the gradient of the curve is never less than $-\frac{16}{3}$. (3)

Solution

We complete the square:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 2x - 5\\ &= 3[x^2 - \tfrac{2}{3}x] - 5\\ &= 3[x^2 - \tfrac{2}{3}x + \tfrac{1}{9} - \tfrac{1}{9}] - 5\\ &= 3[(x - \tfrac{1}{3})^2 - \tfrac{1}{9}] - 5\\ &= 3(x - \tfrac{1}{3})^2 - \tfrac{1}{3} - 5\\ &= 3(x - \tfrac{1}{3})^2 - \tfrac{16}{3}\\ &\geqslant \underline{\underline{-\tfrac{16}{3}}}\end{aligned}$$

when $x = \frac{1}{3}$.

11. Each day a newsagent sells copies of 10 different newspapers, one of which is *The Times*.

A customer buys 3 different newspapers.

Calculate the number of ways the customer can select his newspapers

(a) (i) if there is no restriction, (1)

Solution

$$\begin{aligned}\text{No restrictions} &= \binom{10}{3}\\ &= \underline{\underline{120}}.\end{aligned}$$

(ii) if 1 of the 3 newspapers is *The Times*. (1)

Solution

$$\begin{aligned}\textit{The Times} &= \binom{9}{2}\\ &= \underline{\underline{36}}.\end{aligned}$$

(b) (i) Calculate the number of different 5-digit numbers which can be formed using the digits 0, 1, 2, 3, and 4 without repetition and assuming that a number cannot begin with 0. (2)

Solution

$$\text{Cannot begin with } 0 = 4 \times 4! = \underline{\underline{96}}.$$

(ii) How many of these 5-digit numbers are even? (4)

Solution

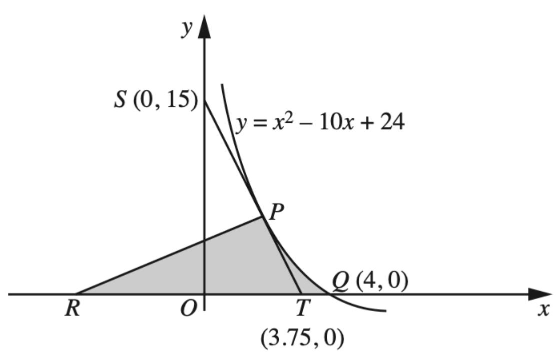
$$\begin{aligned} \text{Even} &= \text{starts 1, even} + \text{starts 2, even} + \text{starts 3, even} + \text{starts 4, even} \\ &= (3 \times 1 \times 3!) + (2 \times 1 \times 3!) + (3 \times 1 \times 3!) + (2 \times 1 \times 3!) \\ &= 18 + 12 + 18 + 12 \\ &= \underline{\underline{60}}. \end{aligned}$$

EITHER

12. The diagram, which is not drawn to scale, shows part of the curve

$$y = x^2 - 10x + 24$$

cutting the x-axis at $Q(4, 0)$.



The tangent to the curve at the point P on the curve meets the coordinate axes at $S(0, 15)$ and at $T(3.75, 0)$.

(a) Find the coordinates of P. (4)

Solution
Now,

$$\begin{aligned} m_{ST} &= \frac{15-0}{0-3.75} \\ &= -4 \end{aligned}$$

and the equation of the line joining S and T is

$$y = -4x + 15.$$

Now,

$$\begin{aligned} x^2 - 10x + 24 = -4x + 15 &\Rightarrow x^2 - 6x + 9 = 0 \\ &\Rightarrow (x-3)^2 = 0 \\ &\Rightarrow x = 3 \\ &\Rightarrow y = 3; \end{aligned}$$

hence, $\underline{\underline{P(3,3)}}$.

The normal to the curve at P meets the x-axis at R.

(b) Find the coordinates of R. (2)

Solution
Well,

$$y = x^2 - 10x + 24 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 10.$$

Now,

$$\begin{aligned} x = 3 &\Rightarrow m_{\text{tangent}} = -4 \\ &\Rightarrow m_{\text{normal}} = -\tfrac{1}{-4} \\ &\Rightarrow m_{\text{normal}} = \tfrac{1}{4} \end{aligned}$$

and the equation of the line joining R and P is

$$\begin{aligned} y - 3 = \tfrac{1}{4}(x-3) &\Rightarrow y - 3 = \tfrac{1}{4}x - \tfrac{3}{4} \\ &\Rightarrow y = \tfrac{1}{4}x - \tfrac{9}{4}. \end{aligned}$$

Next,

$$\begin{aligned} y = 0 &\Rightarrow \tfrac{1}{4}x + \tfrac{9}{4} = 0 \\ &\Rightarrow \tfrac{1}{4}x = -\tfrac{9}{4} \\ &\Rightarrow x = -9; \end{aligned}$$

hence, $\underline{\underline{R(-9, 0)}}$.

(c) Calculate the area of the shaded region bounded by the x-axis, the line PR, and the curve PQ. (5)

Solution

Let $S(3, 0)$. Then

$$\begin{aligned} \text{shaded region} &= \text{triangle } PSR + \int_3^4 (x^2 - 10x + 24)\,\mathrm{d}x \\ &= (\tfrac{1}{2} \times [3 - (-9)] \times 3) + \left[\tfrac{1}{3}x^3 - 5x^2 + 24x\right]_{x=3}^{4} \\ &= 36 + \left(\tfrac{64}{3} - 80 + 96\right) - (9 - 45 + 72) \\ &= 36 + 1\tfrac{1}{3} \\ &= \underline{\underline{37\tfrac{1}{3}}}. \end{aligned}$$

OR

13. A curve has the equation

$$y = 2\cos x - \cos 2x,$$

where $0 < x \leqslant \frac{1}{2}\pi$.

(a) Obtain expressions for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (4)

Solution

$$\begin{aligned} y = 2\cos x - \cos 2x &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin x - 2\sin 2x}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\cos x - 4\cos 2x}}. \end{aligned}$$

(b) Given that $\sin 2x$ may be expressed as $2\sin x\cos x$, find the x-coordinate of the stationary point of the curve and determine the nature of this stationary point. (4)

Solution

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow -2\sin x - 2\sin 2x = 0\\ &\Rightarrow -2\sin x - 2(2\sin x\cos x) = 0\\ &\Rightarrow -2\sin x - 4\sin x\cos x = 0\\ &\Rightarrow -2\sin x(1 - 2\cos x) = 0\\ &\Rightarrow \sin x = 0 \text{ or } \cos x = \tfrac{1}{2}.\end{aligned}$$

Now, $0 < x \leqslant \frac{1}{2}\pi$ which rules $\sin x = 0$ out so we are left with

$$\cos x = \tfrac{1}{2} \Rightarrow \underline{\underline{x = \tfrac{1}{3}\pi}}$$

and

$$x = \tfrac{1}{3}\pi \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 > 0$$

and we conclude that is a <u>minimum</u>.

(c) Evaluate (3)

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} y\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} y\,\mathrm{d}x &= \left[2\sin x - \tfrac{1}{2}\sin 2x\right]_{x=\frac{1}{3}\pi}^{\frac{1}{2}\pi}\\ &= (2 - 0) - \left(\sqrt{3} - \tfrac{1}{4}\sqrt{3}\right)\\ &= \underline{\underline{2 - \tfrac{3}{4}\sqrt{3}}}.\end{aligned}$$