Dr Oliver Mathematics
Applied Mathematics: Integration

The total number of marks available is 103.
You must write down all the stages in your working.

1. (a) Express (3)

$$\frac{x^2+3}{x(1+x^2)}$$

in partial fractions.

(b) Hence obtain (3)

$$\int_{\frac{1}{2}}^{1} \frac{x^2+3}{x(1+x^2)}\,\mathrm{d}x.$$

2. Use the substitution $u = 1 + x^2$ to obtain (5)

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

3. (a) Evaluate (4)

$$\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x.$$

(b) Use part (a) to evaluate (3)

$$\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x.$$

(c) Hence obtain (2)

$$\int_0^1 (3x^2+2x)\mathrm{e}^{2x}\,\mathrm{d}x.$$

4. Find the exact value of (5)

$$\int_0^{\frac{1}{6}\pi} x\sin 3x\,\mathrm{d}x.$$

5. (a) Express (4)

$$\frac{8}{x(x+2)(x+4)}$$

in partial fractions.

(b) Calculate the area under the curve (5)

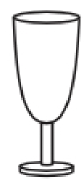
$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$.
Express your answer in the form $\ln \frac{a}{b}$, where a and b are positive integers.

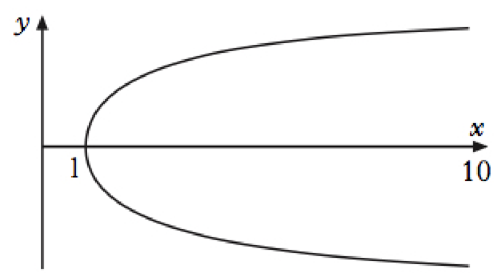
6. (a) Use integration by parts to show that (2)

$$\int \ln x \, \mathrm{d}x = x \ln x - x + c.$$

A goblet consists of a bowl and a short stem.



The diagram below shows the bowl section of the goblet (on its side).



The equation of the upper half of the curve is

$$y = 2\sqrt{\ln x}$$

for $1 \leqslant x \leqslant 10$.

(b) Given that the stem has length 1 and the overall height is 10, what is the capacity of the bowl? (4)

7. Newton's law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22°C, the temperature T°C of a body after a time t minutes satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 22),$$

where k is a negative constant.

(a) Hence show that T can be expressed in the form (4)

$$T = Ae^{kt} + 22$$

for some arbitrary constant A.

In a restaurant, where the temperature remains constant at 22°C, a freshly baked roll, with temperature 82°C, is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20°C.

(b) (i) Calculate the values of A and k. (2)

(ii) Write down an expression for the temperature of the roll after t minutes. (2)

(iii) Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be? (1)

8. Obtain (4)

$$\int_0^{\frac{1}{3}\pi} \cos^5 x \sin x \, dx$$

by using the substitution $u = \cos x$ or otherwise.

9. (a) Express (3)

$$\frac{3x}{(x+1)^2}$$

in partial fractions.

(b) Hence obtain (2)

$$\int \frac{3x}{(x+1)^2} \, dx.$$

10. Use the substitution $u = \ln x$ to obtain (4)

$$\int \frac{2}{x \ln x} \, dx,$$

where $x > 1$.

11. (a) Express (3)

$$\frac{1}{x^2 + x}$$

in partial fractions, where x is neither 0 nor -1.

A region is enclosed by the curve with equation

$$y = \frac{1}{\sqrt{x^2 + x}},$$

the x-axis, and the lines $x = 1$ and $x = 3$.

(b) Calculate the volume of the solid of revolution formed by rotating this region through 360° about the x-axis. (4)

12. Use integration by parts to obtain (4)

$$\int \frac{\ln x}{x^3}\,\mathrm{d}x,$$

where $x > 0$.

13. Find the exact value of (5)

$$\int_0^{2\pi} x \sin 3x\,\mathrm{d}x.$$

14. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton. The situation can be modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000},$$

where N is the number of people infected after t days and $0 < N < 20\,000$.

(a) How many people are infected when the infection is spreading most rapidly? (1)

(b) Express (5)

$$\frac{10\,000}{N(20\,000 - N)}$$

in partial fractions and show that

$$\ln\left(\frac{N}{20\,000 - N}\right) = 2t + c,$$

for some constant c.

Initially there were 100 people infected.

(c) Show that (4)

$$N = \frac{20\,000\,\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}.$$

15. Find the general solution, in the form $y = \mathrm{f}(x)$, of the differential equation (6)

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \tan x,\ 0 < x < \pi.$$

16. (a) Express (3)

$$\frac{1}{1 - y^2}$$

in partial fractions.

(b) Use the substitution $u = \sqrt{1 - x}$ to obtain (6)

$$\int \frac{1}{x\sqrt{1 - x}}\,\mathrm{d}x,\ 0 < x < 1.$$