Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 June Paper 2 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Without using a calculator, express (4)

$$\frac{(5+2\sqrt{3})^2}{2+\sqrt{3}}$$

in the form

$$p+q\sqrt{3},$$

where p and q are integers

Solution

Well,

$\times$	5	$+2\sqrt{3}$
5	25	$+10\sqrt{3}$
$+2\sqrt{3}$	$+10\sqrt{3}$	$+12$

and so

$$\begin{aligned}\frac{(5+2\sqrt{3})^2}{2+\sqrt{3}} &= \frac{37+20\sqrt{3}}{2+\sqrt{3}} \\ &= \frac{37+20\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}\end{aligned}$$

$\times$	37	$+20\sqrt{3}$
2	74	$+40\sqrt{3}$
$-\sqrt{3}$	$-37\sqrt{3}$	-60

$\times$	2	$+\sqrt{3}$
2	4	$+2\sqrt{3}$
$-\sqrt{3}$	$-2\sqrt{3}$	-3

$$= \underline{\underline{14 + 3\sqrt{3}}};$$

hence, $\underline{\underline{p = 14}}$ and $\underline{\underline{q = 3}}$.

2. (a) Find the coefficient of x^3 in the expansion of (2)

$$(1 - \tfrac{1}{2}x)^{12}.$$

Solution

Now, reading ahead to part (b), we want the coefficient of **both** the x^2 and x^3 terms:

$$\begin{aligned}(1 - \tfrac{1}{2}x)^{12} &= \ldots + \binom{12}{2}(-\tfrac{1}{2}x)^2 + \binom{12}{3}(-\tfrac{1}{2}x)^3 + \ldots \\ &= \ldots + \tfrac{33}{2}x^2 - \tfrac{55}{2}x^3 + \ldots;\end{aligned}$$

hence, the coefficient of x^3 is $\underline{\underline{-\tfrac{55}{2}}}$.

(b) Find the coefficient of x^3 in the expansion of (3)

$$(1 + 4x)(1 - \tfrac{1}{2}x)^{12}.$$

Solution

Now,

$\times$	$+\frac{33}{2}x^2$	$-\frac{55}{2}x^3$
1	$\ldots$	$-\frac{55}{2}x^3$
$+4x$	$+66x^3$	$\ldots$

and so the coefficient of x^3 is

$$-\tfrac{55}{2} + 66 = \underline{\underline{\tfrac{77}{2}}}.$$

3. Relative to an origin O, the position vectors of the points A and B are $\mathbf{i} - 4\mathbf{j}$ and $7\mathbf{i} + 20\mathbf{j}$ respectively. (5)

The point C lies on AB and is such that

$$\overrightarrow{AC} = \tfrac{2}{3}\overrightarrow{AB}.$$

Find the position vector of C and the magnitude of this vector.

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 7 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 24 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \tfrac{2}{3}\overrightarrow{AB} \\ &= \begin{pmatrix} 1 \\ -4 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 6 \\ 24 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 4 \\ 16 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 5 \\ 12 \end{pmatrix}}}\end{aligned}$$

and the magnitude of this vector is

$$\sqrt{5^2 + 12^2} = \underline{\underline{13}}.$$

4. Find the set of values of k for which the line (6)

$$y = 2x - 5$$

cuts the curve

$$y = x^2 + kx + 11$$

in two distinct points.

Solution

Now,

$$x^2 + kx + 11 = 2x - 5 \Rightarrow x^2 + (k-2)x + 16 = 0$$

and

$$\begin{aligned} b^2 - 4ac > 0 &\Rightarrow (k-2)^2 - 4(1)(16) > 0 \\ &\Rightarrow (k-2)^2 > 64 \\ &\Rightarrow k - 2 < -8 \text{ or } k - 2 > 8 \\ &\Rightarrow \underline{\underline{k < -6 \text{ or } k > 10}}. \end{aligned}$$

5. The expression

$$x^3 + 8x^2 + px - 25$$

leaves a remainder of R when divided by $(x-1)$ and a remainder of $-R$ when divided by $(x+2)$.

(a) Find the value of p. (4)

Solution

We use synthetic division twice:

1	1	8	p	-25
	$\downarrow$	1	9	$p+9$
	1	9	$p+9$	$p-16$

and

-2	1	8	p	-25
	$\downarrow$	-2	-12	$-2p+24$
	1	6	$p-12$	$-2p-1$

Now,

$$\begin{aligned} p - 16 = -(-2p - 1) &\Rightarrow p - 16 = 2p + 1 \\ &\Rightarrow \underline{\underline{p = -17}}. \end{aligned}$$

(b) Hence find the remainder when the expression is divided by $(x + 3)$. (2)

Solution

$$\begin{array}{r|rrrr} -3 & 1 & 8 & -17 & -25 \\ & \downarrow & -3 & -15 & 96 \\ \hline & 1 & 5 & -32 & 71 \end{array}$$

Hence, the remainder is $\underline{\underline{71}}$.

6. (a) A shelf contains 8 different travel books, of which 5 are about Europe and 3 are about Africa.

(i) Find the number of different ways the books can be arranged if there are no restrictions. (2)

Solution

$$8! = \underline{\underline{40\,320}}.$$

(ii) Find the number of different ways the books can be arranged if the 5 books about Europe are kept together. (2)

Solution

$$4! \times 5! = \underline{\underline{2\,880}}.$$

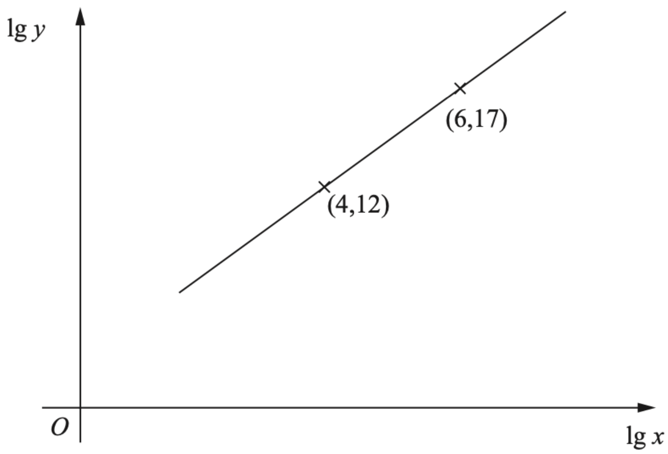
(b) 3 DVDs and 2 videotapes are to be selected from a collection of 7 DVDs and 5 videotapes. (3)

Calculate the number of different selections that could be made.

Solution

$$\begin{aligned} \binom{7}{3} \times \binom{5}{2} &= 35 \times 10 \\ &= \underline{\underline{350}}. \end{aligned}$$

7. The variables x and y are related so that when $\log_{10} y$ is plotted against $\log_{10} x$ a straight line graph passing through the points $(4, 12)$ and $(6, 17)$ is obtained.



(a) Express y in terms of x, giving your answer in the form $y = ax^b$. (6)

Solution
Well,

$$m = \frac{17-12}{6-4} = \tfrac{5}{2}$$

and the equation is

$$\begin{aligned}\log_{10} y - 17 = \tfrac{5}{2}(\log_{10} x - 6) &\Rightarrow \log_{10} y - 17 = \tfrac{5}{2}\log_{10} x - 15 \\ &\Rightarrow \log_{10} y - \tfrac{5}{2}\log_{10} x = 2 \\ &\Rightarrow \log_{10} y - \log_{10} x^{\frac{5}{2}} = 2 \\ &\Rightarrow \log_{10}\left(\frac{y}{x^{\frac{5}{2}}}\right) = 2 \\ &\Rightarrow \frac{y}{x^{\frac{5}{2}}} = 10^2 \\ &\Rightarrow \underline{\underline{y = 100x^{\frac{5}{2}}}};\end{aligned}$$

hence, $\underline{\underline{a = 100}}$ and $\underline{\underline{b = \tfrac{5}{2}}}$.

(b) Find the value of x when $y = 300$. (2)

Solution

$$\begin{aligned}100x^{\frac{5}{2}} = 300 \Rightarrow x^{\frac{5}{2}} &= 3 \\ \Rightarrow x &= 3^{\frac{2}{5}} \\ \Rightarrow x &= 1.551\,845\,574 \text{ (FCD)} \\ \Rightarrow \underline{\underline{x = 1.55 \text{ (3 sf)}}}&.\end{aligned}$$

8. The temperature, T° Celsius, of an object, t minutes after it is removed from a heat source, is given by

$$T = 55\mathrm{e}^{-0.1t} + 15.$$

(a) Find the temperature of the object at the instant it is removed from the heat source. (1)

Solution

$$55 + 15 = \underline{\underline{70^\circ \text{ Celsius}}}.$$

(b) Find the temperature of the object when $t = 8$. (1)

Solution

$$\begin{aligned}t = 8 \Rightarrow T &= 39.713\,093\,03 \text{ (FCD)} \\ \Rightarrow \underline{\underline{T = 39.7^\circ \text{ Celsius (3 sf)}}}&.\end{aligned}$$

(c) Find the value of t when $T = 25$. (3)

Solution

$$\begin{aligned}T = 25 &\Rightarrow 55\mathrm{e}^{-0.1t} + 15 = 25 \\ &\Rightarrow 55\mathrm{e}^{-0.1t} = 10 \\ &\Rightarrow \mathrm{e}^{-0.1t} = \tfrac{2}{11} \\ &\Rightarrow \mathrm{e}^{0.1t} = \tfrac{11}{2} \\ &\Rightarrow 0.1t = \ln \tfrac{11}{2} \\ &\Rightarrow t = 10 \ln \tfrac{11}{2} \\ &\Rightarrow t = 17.047\,480\,92 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 17.0 \text{ minutes (3 sf)}}}.\end{aligned}$$

(d) Find the rate of change of T when $t = 16$. (3)

Solution
Well,

$$T = 55\mathrm{e}^{-0.1t} + 15 \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}t} = -5.5\mathrm{e}^{-0.1t}$$

and

$$\begin{aligned} t = 16 \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}t} &= -1.110\,430\,849 \text{ (FCD)} \\ \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}t} &= -1.11^\circ \text{ Celsius/minutes (3 sf).} \end{aligned}$$

9. A coastguard station receives a distress call from a ship which is travelling at 15 kmh^{-1} on a bearing of 150°.

A lifeboat leaves the coastguard station at 15 : 00 hours; at this time the ship is at a distance of 30 km on a bearing of 270°.

The lifeboat travels in a straight line at constant speed and reaches the ship at 15 : 40 hours.

(a) Find the speed of the lifeboat. (5)

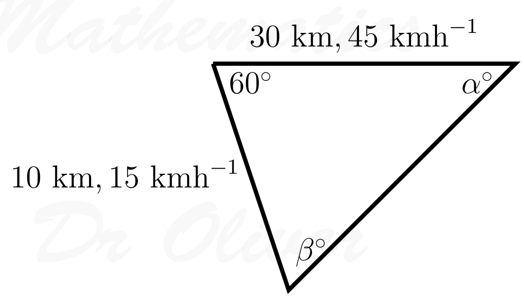
Solution
The ship travels

$$15 \times \tfrac{2}{3} = 10 \text{ km}$$

and the lifeboat is going at

$$\frac{30}{\frac{2}{3}} = 45 \text{ kmh}^{-1}.$$

We draw a sketch:



Cosine rule:

$$v^2 = 45^2 + 15^2 - 2 \times 45 \times 15 \times \cos 60^\circ \Rightarrow v^2 = 1\,575$$
$$\Rightarrow \underline{\underline{v = 15\sqrt{7} \text{ kmh}^{-1}}}.$$

(b) Find the bearing on which the lifeboat travelled. (3)

Solution
Sine rule:

$$\frac{\sin \alpha^\circ}{15} = \frac{\sin 60^\circ}{15\sqrt{7}} \Rightarrow \sin \alpha^\circ = \frac{15 \sin 60^\circ}{15\sqrt{7}}$$
$$\Rightarrow \alpha = 19.106\,605\,35 \text{ (FCD)}.$$

Finally, the bearing on which the lifeboat travelled is

$$270 - 19.106\ldots = 250.893\,394\,6 \text{ (FCD)}$$
$$= \underline{\underline{251^\circ \text{ (3 sf)}}}.$$

10. (a) Solve the equation (3)

$$3 \sin x + 4 \cos x = 0, \text{ for } 0^\circ < x < 360^\circ.$$

Solution
Well,

$$3 \sin x + 4 \cos x = 0 \Rightarrow 3 \sin x = -4 \cos x$$
$$\Rightarrow \tan x = -\tfrac{4}{3}$$
$$\Rightarrow x = 126.869\,897\,6,\ 306.869\,897\,6 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 127,\ 307 \text{ (3 sf)}}}.$$

(b) Solve the equation (5)

$$6 \cos y + 6 \sec y = 13, \text{ for } 0^\circ < y < 360^\circ.$$

Solution

Multiply by $\cos y$:

$$\begin{aligned} 6\cos y + 6\sec y = 13 &\Rightarrow 6\cos^2 y + 6 = 13\cos y \\ &\Rightarrow 6\cos^2 y - 13\cos y + 6 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -13 \\ \text{multiply to:} & (+6)\times(+6) = +36 \end{array}\right\} -9,\ -4$$

e.g.,

$$\begin{aligned} &\Rightarrow 6\cos^2 y - 9\cos y - 4\cos y + 6 = 0 \\ &\Rightarrow 3\cos y(2\cos y - 3) - 2(2\cos y - 3) = 0 \\ &\Rightarrow (3\cos y - 2)(2\cos y - 3) = 0 \\ &\Rightarrow 3\cos y - 2 = 0 \text{ or } 2\cos y - 3 = 0 \\ &\Rightarrow \cos y = \tfrac{2}{3} \text{ or } \cos y = \tfrac{3}{2} \text{ (can't happen!)} \\ &\Rightarrow y = 48.189\,685\,1,\ 311.810\,314\,9 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{y = 48.2,\ 312 \text{ (3 sf)}}}. \end{aligned}$$

(c) Solve the equation (3)

$$\sin(2z - 3) = 0.7, \text{ for } 0 < z < \pi \text{ radians.}$$

Solution

Well,

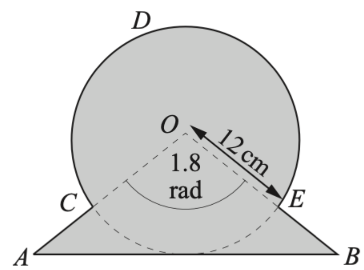
$$\begin{aligned} 0 < z < \pi &\Rightarrow 0 < 2z < 2\pi \\ &\Rightarrow -3 < 2z - 3 < 3.283\ldots \end{aligned}$$

and

$$\begin{aligned} & \sin(2z-3) = 0.7 \\ \Rightarrow\ & 2z - 3 = 0.775\,397\,496\,6,\ 2.366\,195\,157 \text{ (FCD)} \\ \Rightarrow\ & 2z = 3.775\,397\,496\,6, 5.66\,195\,157 \text{ (FCD)} \\ \Rightarrow\ & z = 1.887\,698\,748,\ 2.683\,097\,75 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{z = 1.89,\ 2.68 \text{ (3 sf)}}}. \end{aligned}$$

EITHER

11. The diagram shows an isosceles triangle AOB and a sector $OCDEO$ of a circle with centre O.



- The line AB is a tangent to the circle.
- Angle $AOB = 1.8$ radians and the radius of the circle is 12 cm.

(a) Show that the distance $AC = 7.3$ cm to 1 decimal place. (2)

Solution

Let F be the midpoint of AB. Then

$$\begin{aligned}\cos = \frac{\text{adj}}{\text{hyp}} &\Rightarrow \cos 0.9 = \frac{12}{OA} \\ &\Rightarrow OA = \frac{12}{\cos 0.9} \\ &\Rightarrow AC = \frac{12}{\cos 0.9} - 12 \\ &\Rightarrow AC = 7.304\,709\,726 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{AC = 7.3 \text{ cm (1 dp)}}}.\end{aligned}$$

(b) Find the perimeter of the shaded region. (6)

Solution

Well,

$$\begin{aligned}\text{perimeter} &= \text{arc } CDE + EB + AB + AC \\ &= [12 \times (2\pi - 1.8)] + 7.304\ldots + 2AF + 7.304\ldots \\ &= [12 \times (2\pi - 1.8)] + 7.304\ldots + (2 \times 12 \tan 0.9) + 7.304\ldots \\ &= 98.651\,440\,36 \text{ (FCD)} \\ &= \underline{\underline{98.7 \text{ cm (3 sf)}}}.\end{aligned}$$

(c) Find the area of the shaded region. (4)

Solution

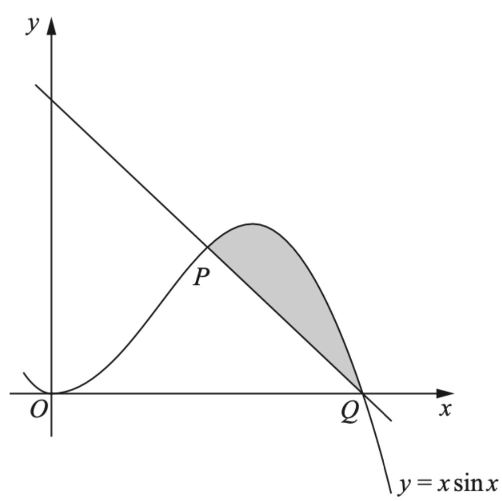
$$\begin{aligned}\text{Shaded region} &= \text{area } CDEOC + \text{area } OAB \\ &= [\tfrac{1}{2} \times 12^2 \times (2\pi - 1.8)] + \left[\tfrac{1}{2} \times \left(\frac{12}{\cos 0.9}\right)^2 \times \sin 1.8\right] \\ &= 504.252\,125\,4 \text{ (FCD)} \\ &= \underline{\underline{504 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

OR

12. The diagram shows part of the curve

$$y = x \sin x$$

and the normal to the curve at the point $P(\frac{1}{2}\pi, \frac{1}{2}\pi)$.



The curve passes through the point $Q(\pi, 0)$.

(a) Show that the normal to the curve at P passes through the point Q. (4)

Solution

Product rule:

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos + \sin x.$$

Now,

$$x = \tfrac{1}{2}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and

$$m_{\text{normal}} = -1.$$

Next, the normal to the curve is

$$y - \tfrac{1}{2}\pi = -(x - \tfrac{1}{2}\pi)$$

and

$$y = 0 \Rightarrow x = \pi;$$

so, the normal to the curve at P passes through the point Q.

(b) Given that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\cos x) = \cos x - x\sin x,$$

find

$$\int x\sin x\,\mathrm{d}x.$$

Solution

Well,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x\cos x) = \cos x - x\sin x \Rightarrow x\cos x &= \int(\cos x - x\sin x)\,\mathrm{d}x\\ \Rightarrow x\cos x &= \int\cos x\,\mathrm{d}x - \int x\sin x\,\mathrm{d}x\\ \Rightarrow \int x\sin x\,\mathrm{d}x &= \sin x - x\cos x + c.\end{aligned}$$

(c) Find the area of the shaded region. (5)

Solution

Let $R(\frac{1}{2}\pi, 0)$. Then

$$\begin{aligned}
\text{shaded region} &= \int_{\frac{1}{2}\pi}^{\pi} \sin x \, \mathrm{d}x - \text{area of } PQR \\
&= \left[\sin x - x\cos x\right]_{x=\frac{1}{2}\pi}^{\pi} - \left(\tfrac{1}{2} \times \tfrac{1}{2}\pi \times \tfrac{1}{2}\pi\right) \\
&= (0+\pi) - (1-0) - \tfrac{1}{8}\pi^2 \\
&= \underline{\underline{\pi - \tfrac{1}{8}\pi^2 - 1}}.
\end{aligned}$$