Dr Oliver Mathematics
Further Mathematics
Complex Numbers: de Moivre's Theorem
Past Examination Questions

This booklet consists of 12 questions across a variety of examination topics.
The total number of marks available is 130.

Let $s = \sin\theta$, $c = \cos\theta$, and $t = \tan\theta$. Then

$$\begin{aligned}(1 + \mathrm{i}t)^n &\equiv \left[\tfrac{1}{c}(c + \mathrm{i}s)\right]^n \\ &\equiv \frac{1}{c^n}(c + \mathrm{i}s)^n \\ &\equiv \frac{1}{c^n}\left[(\text{real part}) + \mathrm{i}(\text{imaginary part})\right]\end{aligned}$$

and we have

$$\tan n\theta \equiv \frac{\frac{1}{c^n}(\text{imaginary part})}{\frac{1}{c^n}(\text{real part})} \equiv \frac{\text{imaginary part}}{\text{real part}}.$$

For example, use de Moivre's theorem to show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

Now,

$$\begin{aligned}(1 + \mathrm{i}t)^4 &\equiv 1 + 4(\mathrm{i}t) + 6(\mathrm{i}t)^2 + 4(\mathrm{i}t)^3 + (\mathrm{i}t)^4 \\ &\equiv 1 - 6t^2 + t^4 + \mathrm{i}(4t - 4t^3)\end{aligned}$$

and we have

$$\tan 4\theta \equiv \frac{\text{imaginary part}}{\text{real part}} \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

1. (a) Given that $z = \mathrm{e}^{\mathrm{i}\theta}$, show that (2)

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

where n is a positive integer.

Solution

$$z^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

and

$$z^{-n} = \left(e^{i\theta}\right)^{-n} = e^{-in\theta} = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta.$$

Hence

$$\underline{\underline{z^n - z^{-n} = 2i \sin n\theta}},$$

as required.

(b) Show that (5)

$$\sin^5 \theta = \tfrac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta).$$

Solution

$$\begin{aligned}
\sin^5 \theta &= \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^5 \\
&= \frac{1}{32i}\left[z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}\right] \\
&= \frac{1}{16}\left[\frac{1}{2i}\left(z^5 - \frac{1}{z^5}\right) - \frac{5}{2i}\left(z^3 - \frac{1}{z^3}\right) + \frac{10}{2}\left(z - \frac{1}{z}\right)\right] \\
&= \underline{\underline{\frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)}}.
\end{aligned}$$

(c) Hence solve, in the interval $0 \leqslant \theta < 2\pi$, (5)

$$\sin 5\theta - 5 \sin 3\theta + 6 \sin \theta = 0.$$

Solution

$$\begin{aligned}
& \sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0 \\
\Rightarrow\quad & \sin 5\theta - 5\sin 3\theta + 10\sin\theta = 4\sin\theta \\
\Rightarrow\quad & 16\sin^5\theta = 4\sin\theta \\
\Rightarrow\quad & 16\sin^5\theta - 4\sin\theta = 0 \\
\Rightarrow\quad & 4\sin\theta(4\sin^4\theta - 1) = 0 \\
\Rightarrow\quad & \sin\theta = 0 \text{ or } \sin\theta = \pm\tfrac{\sqrt{2}}{2}.
\end{aligned}$$

$\underline{\sin\theta = 0}$: $\sin\theta = 0 \Rightarrow \theta = \underline{\underline{0, \pi}}$.

$\underline{\sin\theta = \frac{\sqrt{2}}{2}}$: $\sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \underline{\underline{\frac{\pi}{4}, \frac{3\pi}{4}}}$.

$\underline{\sin\theta = -\frac{\sqrt{2}}{2}}$: $\sin\theta = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \underline{\underline{\frac{5\pi}{4}, \frac{7\pi}{4}}}$.

2. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned}
\cos 5\theta + \mathrm{i}\sin 5\theta &\equiv (c + \mathrm{i}s)^5 \\
&\equiv c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5 \\
&\equiv c^5 - 10c^3s^2 + 5cs^4 + \mathrm{i}(5c^4s - 10c^2s^3 + s^5).
\end{aligned}$$

Comparing the imaginary parts,

$$\begin{aligned}
\sin 5\theta &\equiv 5c^4s - 10c^2s^3 + s^5 \\
&= s(5c^4 - 10c^2s^2 + s^4) \\
&= s\left[5c^4 - 10c^2(1 - c^2)^2 + (1 - c^2)^2\right] \\
&= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4) \\
&= \underline{\underline{\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)}},
\end{aligned}$$

as required.

(b) Hence, or otherwise, solve, for $0 \leqslant \theta < \pi$, (6)

$$\sin 5\theta + \cos\theta\sin 2\theta = 0.$$

Solution

$$\begin{aligned} & \sin 5\theta + \cos\theta \sin 2\theta = 0 \\ \Rightarrow\quad & \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\cos^2\theta\sin\theta = 0 \\ \Rightarrow\quad & \sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0 \\ \Rightarrow\quad & \sin\theta(8\cos^2\theta - 1)(2\cos^2\theta - 1) = 0 \\ \Rightarrow\quad & \sin\theta = 0, \cos\theta = \pm\tfrac{\sqrt{2}}{4}, \text{ or } \cos\theta = \pm\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$\underline{\sin\theta = 0}$: $\sin\theta = 0 \Rightarrow \underline{\underline{\theta = 0}}$.

$\underline{\cos\theta = \pm\frac{\sqrt{2}}{4}}$: $\cos\theta = \pm\frac{\sqrt{2}}{4} \Rightarrow \theta = 1.209\,429\,203, 1.932\,163\,451$ (FCD).

$\underline{\cos\theta = \pm\frac{\sqrt{2}}{2}}$: $\cos\theta = \pm\frac{\sqrt{2}}{2} \Rightarrow \underline{\underline{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}}$.

3. (a) Use de Moivre's theorem to show that (2)

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

Solution

$$z^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

and

$$z^{-n} = \left(e^{i\theta}\right)^{-n} = e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$

Hence

$$\underline{\underline{z^n + z^{-n} = 2\cos n\theta}},$$

as required.

(b) Express $32\cos^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where p, q, r, s are integers. (5)

Solution

Using part (a),

$$\begin{aligned}32\cos^6\theta &= 32\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^6\\ &= \frac{1}{2}\left[z^6+6z^4+15z^2+20+\frac{150}{z^2}+\frac{6}{z^4}+\frac{1}{z^6}\right]\\ &= \frac{1}{2}\left(z^6+\frac{1}{z^6}\right)+\frac{6}{2}\left(z^4+\frac{1}{z^4}\right)+\frac{15}{2}\left(z^2+\frac{1}{z^2}\right)+10\\ &= \underline{\underline{\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10}},\end{aligned}$$

as required.

(c) Hence find the exact value of (4)

$$\int_0^{\frac{\pi}{3}}\cos^6\theta\,\mathrm{d}\theta.$$

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\cos^6\theta\,\mathrm{d}\theta &= \int_0^{\frac{\pi}{3}}\left(\tfrac{1}{32}\cos 6\theta+\tfrac{3}{16}\cos 4\theta+\tfrac{15}{32}\cos 2\theta+\tfrac{5}{16}\right)\mathrm{d}\theta\\ &= \left[\tfrac{1}{192}\sin 6\theta+\tfrac{3}{64}\sin 4\theta+\tfrac{15}{64}\sin 2\theta+\tfrac{5}{16}\theta\right]_{\theta=0}^{\frac{\pi}{3}}\\ &= \left(0-\tfrac{3\sqrt{3}}{128}+\tfrac{15\sqrt{3}}{128}+\tfrac{5\pi}{48}\right)-(0+0+0+0)\\ &= \underline{\underline{\tfrac{3\sqrt{3}}{32}+\tfrac{5\pi}{48}}}.\end{aligned}$$

4. De Moivre's theorem states that

$$(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta, \text{ for } n\in\mathbb{R}.$$

(a) Use induction to prove de Moivre's theorem for $n\in\mathbb{Z}^+$. (5)

Solution

$\underline{n=1}$: $\cos\theta+\mathrm{i}\sin\theta$ and $\cos 1\theta+\mathrm{i}\sin 1\theta=\cos\theta+\mathrm{i}\sin\theta$ so we have proved $n=1$.
Suppose the solution is true for $n=k$, i.e.,

$$(\cos\theta+\mathrm{i}\sin\theta)^k=\cos k\theta+\mathrm{i}\sin k\theta.$$

$$
\begin{aligned}
(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos\theta + \mathrm{i}\sin\theta)^{k}(\cos\theta + \mathrm{i}\sin\theta) \\
&= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta) \\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) \\
&\qquad\qquad + \mathrm{i}(\sin k\theta\cos\theta + \sin\theta\cos k\theta) \\
&= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta,
\end{aligned}
$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^{+}$, as required.

(b) Show that (5)

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$
\begin{aligned}
\cos 5\theta + \mathrm{i}\sin 5\theta &\equiv (c + \mathrm{i}s)^5 \\
&\equiv c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5 \\
&\equiv c^5 - 10c^3s^2 + 5cs^4 + \mathrm{i}(5c^4s - 10c^2s^3 + s^5).
\end{aligned}
$$

Comparing the real parts,

$$
\begin{aligned}
\cos 5\theta &\equiv c^5 - 10c^3s^2 + 5cs^4 \\
&\equiv c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2 \\
&\equiv c^5 - 10c^3 + 10c^5 + 5c^3 - 10c^3 + 5c^5 \\
&\equiv \underline{\underline{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}},
\end{aligned}
$$

as required.

(c) Hence show that $2\cos\frac{\pi}{10}$ is a root of the equation (3)

$$x^4 - 5x^2 + 5 = 0.$$

Solution

Let $x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2}$. Then

$$5x^2 = 5(2\cos\theta)^2 = 20\cos^2\theta \text{ and } x^4 = (2\cos\theta)^4 = 16\cos^4\theta.$$

Now,

$$16\cos^4\theta - 20\cos^2\theta + 5 = \frac{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}{\cos\theta} = \frac{\cos 5\theta}{\cos\theta}$$

which means

$$\begin{aligned} 16\cos^4\theta - 20\cos^2\theta + 5 = 0 &\Rightarrow \frac{\cos 5\theta}{\cos\theta} = 0 \\ &\Rightarrow \cos 5\theta = 0 \\ &\Rightarrow 5\theta = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \ldots \end{aligned}$$

We will take the first of these:

$$5\theta = \tfrac{\pi}{2} \Rightarrow \theta = \tfrac{\pi}{10}$$

which makes $\underline{\underline{x = 2\cos\frac{\pi}{10}}}$ a solution.

5. (a) Use de Moivre's theorem to show that (5)

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^6\theta - 1.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned} & \cos 6\theta + \mathrm{i}\sin 5\theta \\ \equiv\ & (c + \mathrm{i}s)^6 \\ \equiv\ & c^6 + 6c^5(\mathrm{i}s) + 15c^4(\mathrm{i}s)^2 + 20c^3(\mathrm{i}s)^3 + 15c^2(\mathrm{i}s)^4 + 6c(\mathrm{i}s)^5 + (\mathrm{i}s)^6 \\ \equiv\ & c^6 - 15c^4s^2 + 15c^2s^4 - s^6 + \mathrm{i}(6c^5s - 20c^3s^3 + 6cs^5). \end{aligned}$$

Comparing the real parts,

$$\begin{aligned} \cos 6\theta &\equiv c^6 - 15c^4s^2 + 15c^2s^4 - s^6 \\ &= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3 \\ &= c^6 - 15c^4(1-c^2) + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6) \\ &= \underline{\underline{32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}}, \end{aligned}$$

as required.

(b) Hence, or otherwise, solve the equation $\cos 6\theta = \cos 2\theta$, $0 \leqslant \theta \leqslant \pi$. (6)

Solution

$$\begin{aligned}\cos 6\theta = \cos 2\theta &\Rightarrow 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 2\cos^2\theta - 1\\ &\Rightarrow 32\cos^6\theta - 48\cos^4\theta + 16\cos^2\theta = 0\\ &\Rightarrow 2\cos^6\theta - 3\cos^4\theta + \cos^2\theta = 0\\ &\Rightarrow \cos^2\theta(2\cos^4\theta - 3\cos^2\theta + 1) = 0\\ &\Rightarrow \cos^2\theta(2\cos^2\theta - 1)(\cos^2\theta - 1) = 0\\ &\Rightarrow \cos\theta = 0, \cos\theta = \pm 1, \text{ or } \cos\theta = \pm\tfrac{\sqrt{2}}{2}.\end{aligned}$$

$\underline{\cos\theta = 0}$: $\cos\theta = 0 \Rightarrow \theta = \underline{\underline{\frac{\pi}{2}}}$.

$\underline{\cos\theta = 1}$: $\cos\theta = 1 \Rightarrow \theta = \underline{\underline{0}}$.

$\underline{\cos\theta = -1}$: $\cos\theta = -1 \Rightarrow \theta = \underline{\underline{\pi}}$.

$\underline{\cos\theta = \frac{\sqrt{2}}{2}}$: $\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \underline{\underline{\frac{\pi}{4}}}$.

$\underline{\cos\theta = -\frac{\sqrt{2}}{2}}$: $\cos\theta = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \underline{\underline{\frac{3\pi}{4}}}$.

6. (a) Use de Moivre's theorem to show that (5)

$$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &\equiv (c + \mathrm{i}s)^5\\ &\equiv c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5\\ &\equiv c^5 - 10c^3s^2 + 5cs^4 + \mathrm{i}(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

Comparing the real parts,

$$\begin{aligned}\cos 5\theta &\equiv c^5 - 10c^3s^2 + 5cs^4\\ &\equiv c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2\\ &\equiv c^5 - 10c^3 + 10c^5 + 5c^3 - 10c^3 + 5c^5\\ &\equiv \underline{\underline{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}},\end{aligned}$$

as required.

(b) Hence find the two positive solutions of (6)

$$32x^5 - 40x^3 + 10x + 1 = 0,$$

giving your answers to 3 decimal places.

Solution

Let $x = \cos\theta$. Then

$$\begin{aligned}32x^5 - 40x^3 + 10x + 1 = 0 &\Rightarrow 32x^5 - 40x^3 + 10x = -1\\ &\Rightarrow 16x^5 - 20x^3 + 5x = -\tfrac{1}{2}\\ &\Rightarrow \cos 5\theta = -\tfrac{1}{2}\\ &\Rightarrow 5\theta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}\\ &\Rightarrow \theta = \tfrac{2\pi}{15}, \tfrac{4\pi}{15}\\ &\Rightarrow x = 0.913\,545\,457\,6, 0.669\,130\,606\,4 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 0.914, 0.669 \text{ (3 dp)}}}.\end{aligned}$$

7. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &\equiv (c + \mathrm{i}s)^5\\ &\equiv c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5\\ &\equiv c^5 - 10c^3s^2 + 5cs^4 + \mathrm{i}(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

Comparing the imaginary parts,

$$\begin{aligned}\sin 5\theta &\equiv 5c^4s - 10c^2s^3 + s^5\\ &= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5\\ &= 5s(1 - 2s^2 + s^4) - 10s^3(1-s^2) + s^5\\ &= \underline{\underline{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}},\end{aligned}$$

as required.

Hence, given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$,

(b) find all the solutions of (6)

$$\sin 5\theta = 5\sin 3\theta,$$

in the interval $0 \leqslant \theta < 2\pi$. Give your answers to 3 decimal places.

Solution

Hence

$$\begin{aligned} &16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 15\sin\theta - 20\sin^3\theta \\ \Rightarrow\ &16\sin^5\theta - 10\sin\theta = 0 \\ \Rightarrow\ &2\sin\theta(8\sin^4\theta - 5) = 0. \end{aligned}$$

Now

$$\sin\theta = 0 \Rightarrow \theta = \underline{\underline{0, \pi}},$$

and

$$\begin{aligned} & 8\sin^4\theta - 5 = 0 \\ \Rightarrow\quad & \sin\theta = \pm\sqrt[4]{\tfrac{5}{8}} \\ \Rightarrow\quad & \theta = 1.095\,461\,855, 2.046\,130\,799, 4.237\,054\,508, 5.188\,723\,452 \text{ (FCD)} \\ \Rightarrow\quad & \theta = \underline{\underline{1.095, 2.046, 4.237, 5.188 \text{ (3 dp)}}}. \end{aligned}$$

8. Given than (5)

$$z = r(\cos\theta + \mathrm{i}\sin\theta), r \in \mathbb{R},$$

prove, by induction, that

$$z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), n \in \mathbb{Z}^+.$$

Solution

$\underline{n = 1}$: $r(\cos\theta + \mathrm{i}\sin\theta)^1 = r(\cos\theta + \mathrm{i}\sin\theta)$ and $r^1(\cos 1\theta + \mathrm{i}\sin 1\theta) = r(\cos\theta + \mathrm{i}\sin\theta)$ so we have proved $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$[r(\cos\theta + \mathrm{i}\sin\theta)]^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta).$$

$$
\begin{aligned}
[r(\cos\theta + \mathrm{i}\sin\theta)]^{k+1} &= [r(\cos\theta + \mathrm{i}\sin\theta)]^k \times r(\cos\theta + \mathrm{i}\sin\theta) \\
&= r^k(\cos k\theta + \mathrm{i}\sin k\theta) \times r(\cos\theta + \mathrm{i}\sin\theta) \\
&= r^{k+1}(\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta) \\
&= r^{k+1}\left[(\cos k\theta\cos\theta - \sin k\theta\sin\theta)\right. \\
&\qquad\qquad \left.+\mathrm{i}(\sin k\theta\cos\theta + \sin\theta\cos k\theta)\right] \\
&= r^{k+1}(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta),
\end{aligned}
$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

9. The complex number $z = e^{i\theta}$, where θ is real.

(a) Use de Moivre's theorem to show that (2)

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

Solution

$$z^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

and

$$z^{-n} = \left(e^{i\theta}\right)^{-n} = e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$

Hence

$$\underline{\underline{z^n + z^{-n} = 2\cos n\theta}},$$

as required.

(b) Show that (5)

$$\cos^5\theta \equiv \tfrac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).$$

Solution

Using part (a),

$$\begin{aligned}\cos^5\theta &= \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^5 \\ &= \frac{1}{32}\left[z^5+5z^3+10z+\frac{10}{z}+\frac{5}{z^3}+\frac{1}{z^5}\right] \\ &= \frac{1}{16}\left[\frac{1}{2}\left(z^5+\frac{1}{z^5}\right)+\frac{5}{2}\left(z^3+\frac{1}{z^3}\right)+\frac{10}{2}\left(z+\frac{1}{z}\right)\right] \\ &= \underline{\underline{\frac{1}{16}(\cos 5\theta+5\cos 3\theta+10\cos\theta)}},\end{aligned}$$

as required.

(c) Hence find all of the solutions of (4)

$$\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$$

in the interval $0 \leqslant \theta < 2\pi$.

Solution

$$\begin{aligned}\cos 5\theta + 5\cos 3\theta + 12\cos\theta &= \cos 5\theta + 5\cos 3\theta + 10\cos\theta + 2\cos\theta \\ &= 16\cos^5\theta + 2\cos\theta \\ &= 2\cos\theta(8\cos^4\theta + 1).\end{aligned}$$

So the desired equation has solutions precisely when $\cos\theta = 0$ and hence the only two solutions in the range are

$$\underline{\underline{\theta = \frac{\pi}{2}, \frac{3\pi}{2}}}.$$

10. (a) Use de Moivre's theorem to show that (5)

$$\cos 5\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned} & \cos 6\theta + \mathrm{i}\sin 6\theta \\ \equiv\ & (c + \mathrm{i}s)^6 \\ \equiv\ & c^6 + 6c^5(\mathrm{i}s) + 15c^4(\mathrm{i}s)^2 + 20c^3(\mathrm{i}s)^3 + 15c^2(\mathrm{i}s)^4 + 6c(\mathrm{i}s)^5 + (\mathrm{i}s)^6 \\ \equiv\ & c^6 - 15c^4s^2 + 15c^2s^4 - s^6 + \mathrm{i}(6c^5s - 20c^3s^3 + 6cs^5). \end{aligned}$$

Comparing the real parts,

$$\begin{aligned} \cos 6\theta &\equiv c^6 - 15c^4s^2 + 15c^2s^4 - s^6 \\ &\equiv c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3 \\ &\equiv c^6 - 15c^4(1-c^2) + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6) \\ &\equiv c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6 \\ &\equiv \underline{\underline{32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}}. \end{aligned}$$

as required.

(b) Hence solve (5)

$$64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0, 0 \leqslant \theta \leqslant \tfrac{\pi}{2},$$

giving your answers as exact multiples of π.

Solution

$$\begin{aligned} & 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0 \\ \Rightarrow\ & 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2 = 1 \\ \Rightarrow\ & 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \tfrac{1}{2} \\ \Rightarrow\ & \cos 6\theta = \tfrac{1}{2} \\ \Rightarrow\ & 6\theta = \tfrac{\pi}{3}, \tfrac{5\pi}{3}, \tfrac{7\pi}{3} \\ \Rightarrow\ & \underline{\underline{\theta = \tfrac{\pi}{18}, \tfrac{5\pi}{18}, \tfrac{7\pi}{18}}}. \end{aligned}$$

11. (a) Use de Moivre's theorem to show that (5)

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Solution
Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &\equiv (c + \mathrm{i}s)^5 \\ &\equiv c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5 \\ &\equiv c^5 - 10c^3s^2 + 5cs^4 + \mathrm{i}(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

Comparing the imaginary parts,

$$\begin{aligned}\sin 5\theta &\equiv 5c^4s - 10c^2s^3 + s^5 \\ &= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5 \\ &= 5s(1 - 2s^2 + s^4) - 10s^3(1-s^2) + s^5 \\ &= \underline{\underline{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}},\end{aligned}$$

as required.

(b) Hence find the five distinct solutions of the equation (5)

$$16x^5 - 20x^3 + 5x + \tfrac{1}{2} = 0,$$

giving your answers to 3 decimal places where necessary.

Solution
Let $x = \sin\theta$. Then

$$\begin{aligned}& 16x^5 - 20x^3 + 5x + \tfrac{1}{2} = 0 \\ \Rightarrow\ & 16x^5 - 20x^3 + 5x = -\tfrac{1}{2} \\ \Rightarrow\ & \sin 5\theta = -\tfrac{1}{2} \\ \Rightarrow\ & 5\theta = 210^\circ, 330^\circ, (570), (690), 930^\circ, 1050^\circ, 1290^\circ \\ \Rightarrow\ & \theta = 42^\circ, 66^\circ, (114), (138), 186^\circ, 210^\circ, 258^\circ \\ \Rightarrow\ & x = 0.669\,130\,606\,4, 0.913\,545\,457\,6, -0.104\,528\,463\,3, \\ & \qquad\qquad -0.5, -0.978\,147\,600\,7 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{x = 0.669, 0.914, -0.105, -0.5, -0.978 \text{ (3 dp)}}}.\end{aligned}$$

because $\sin 66^\circ = \sin 114^\circ$ and $\sin 42^\circ = \sin 138^\circ$.

(c) Use the identity given in (a) to find (4)

$$\int_0^{\frac{\pi}{4}} (4\sin^5\theta - 5\sin^3\theta)\,\mathrm{d}\theta,$$

expressing your answer in the form $a\sqrt{2}+b$, where a and b are rational numbers.

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} (4\sin^5\theta - 5\sin^3\theta)\,\mathrm{d}\theta &= \tfrac{1}{4}\int_0^{\frac{\pi}{4}} (16\sin^5\theta - 20\sin^3\theta)\,\mathrm{d}\theta \\
&= \tfrac{1}{4}\int_0^{\frac{\pi}{4}} (\sin 5\theta - 5\sin\theta)\,\mathrm{d}\theta \\
&= \tfrac{1}{4}\left[-\tfrac{1}{5}\cos 5\theta + 5\cos\theta\right]_{\theta=0}^{\frac{\pi}{4}} \\
&= \tfrac{1}{4}\left[(\tfrac{\sqrt{2}}{10} + \tfrac{5\sqrt{2}}{2}) - (-\tfrac{1}{5} + 5)\right] \\
&= \underline{\underline{\tfrac{13}{20}\sqrt{2} - \tfrac{6}{5}}}.
\end{aligned}$$

12. (a) Use de Moivre's theorem to show that (5)

$$\sin^5\theta \equiv a\sin 5\theta + b\sin 3\theta + c\sin\theta,$$

where a, b, and c are constants to be found.

Solution

Let $c = \cos\theta$ and $s = \sin\theta$. Then

$$\begin{aligned}
\sin^5\theta &= \left[\tfrac{1}{2}(z - \tfrac{1}{z})\right]^5 \\
&= \tfrac{1}{32}\left(z^5 - 5z^3 + 10z - \tfrac{10}{z} + \tfrac{5}{z^3} - \tfrac{1}{z^5}\right) \\
&= \tfrac{1}{16}\left[\tfrac{1}{2}(z^5 - \tfrac{1}{z^5}) - \tfrac{5}{2}(z^3 - \tfrac{1}{z^3}) + \tfrac{10}{2}(z - \tfrac{1}{z})\right] \\
&= \tfrac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right] \\
&= \underline{\underline{\tfrac{1}{16}\sin 5\theta - \tfrac{5}{16}\sin 3\theta + \tfrac{5}{8}\sin\theta}}.
\end{aligned}$$

(b) Hence show that (5)

$$\int_0^{\frac{\pi}{3}} \sin^5\theta\,\mathrm{d}\theta = \tfrac{53}{480}.$$

Solution

$$
\begin{aligned}
\int_0^{\frac{\pi}{3}} \sin^5 \theta \, d\theta &= \int_0^{\frac{\pi}{3}} \left[\tfrac{1}{16}\sin 5\theta - \tfrac{5}{16}\sin 3\theta + \tfrac{5}{8}\sin\theta\right] \, d\theta \\
&= \left[-\tfrac{1}{80}\cos 5\theta + \tfrac{5}{48}\cos 3\theta - \tfrac{5}{8}\cos\theta\right]_{\theta=0}^{\frac{\pi}{3}} \\
&= \left(-\tfrac{1}{160} - \tfrac{5}{48} - \tfrac{5}{16}\right) - \left(-\tfrac{1}{80} + \tfrac{5}{48} - \tfrac{5}{8}\right) \\
&= \underline{\underline{\tfrac{53}{480}}}.
\end{aligned}
$$