Dr Oliver Mathematics
Mathematics
Geometric Series
Past Examination Questions

This booklet consists of 22 questions across a variety of examination topics.
The total number of marks available is 196.

1. The first three terms of a geometric series are

$$18, 12, \text{ and } p$$

respectively, where p is a constant. Find

(a) the value of the common series of the series, (1)

Solution

$$r = \tfrac{12}{18} = \underline{\underline{\tfrac{2}{3}}}.$$

(b) the value of p, (1)

Solution

$$\frac{p}{12} = \frac{12}{18} \Rightarrow \underline{\underline{p = 8}}.$$

(c) the sum of the first 15 terms of the series, giving your answer to 3 decimal places. (2)

Solution

$a = 18, r = \frac{2}{3}, n = 15$:

$$\frac{18\left[1 - \left(\frac{2}{3}\right)^{15}\right]}{1 - \frac{2}{3}} = 53.876\,682\,45 \text{ (FCD)} = \underline{\underline{53.877 \text{ (3 dp)}}}.$$

2. A geometric series has first term $a = 360$ and common ratio $r = \frac{7}{8}$. Giving your answer to 3 significant figures where appropriate, find

(a) the 20th term of the series, (2)

Solution

$$20\text{th term} = 360 \times \left(\tfrac{7}{8}\right)^{19} = 28.474\,460\,075 \text{ (FCD)} = \underline{\underline{28.5 \text{ (3 sf)}}}.$$

(b) the sum of the first 20 terms of the series, (2)

Solution

$a = 360, r = \frac{7}{8}, n = 20$:

$$\frac{360\left[1 - \left(\frac{7}{8}\right)^{20}\right]}{1 - \frac{7}{8}} = 2680.678\,775 \text{ (FCD)} = \underline{\underline{2680 \text{ (3 sf)}}}.$$

(c) the sum to infinity of the series. (2)

Solution

$$\frac{360}{1 - \frac{7}{8}} = \underline{\underline{2880}}.$$

3. The fourth term of a geometric series is 10 and the seventh term of the series is 80. For this series, find

(a) the common ratio, (2)

Solution

$$r^3 = \frac{80}{10} = 8 \Rightarrow \underline{\underline{r = 2}}.$$

(b) the first term, (2)

Solution

$$a = 10 \div 2^3 = \underline{\underline{1.25}}.$$

(c) the sum of the first 20 terms, giving your answer to the nearest whole number. (2)

Solution

$$\frac{1.25\left[1 - 2^{20}\right]}{1 - 2} = 1\,310\,718.75 \text{ (FCD)} = \underline{\underline{1\,310\,719 \text{ (nearest whole number)}}}.$$

4. A geometric series has first term a and common ratio $r = \frac{3}{4}$. The sum of the first four terms of this series is 175.

(a) Show that $a = 64$. (2)

Solution

$$a + \tfrac{3}{4}a + \left(\tfrac{3}{4}\right)^2 a + \left(\tfrac{3}{4}\right)^3 a = 175 \Rightarrow \tfrac{175}{64}a = 175 \Rightarrow \underline{\underline{a = 64}}.$$

(b) Find the sum to infinity of the series. (2)

Solution

$$\frac{64}{1 - \frac{3}{4}} = \underline{\underline{256}}.$$

(c) Find the difference between the 9th and 10th terms of the series. Give your answer to 3 decimal places. (3)

Solution

$$\begin{aligned} \text{9th term - 10th term} &= 64\left(\tfrac{3}{4}\right)^8 - 64\left(\tfrac{3}{4}\right)^9 \\ &= 64\left(\tfrac{3}{4}\right)^8\left[1 - \tfrac{3}{4}\right] \\ &= 1.601\,806\,641 \text{ (FCD)} \\ &= \underline{\underline{1.602 \text{ (3 dp)}}}. \end{aligned}$$

5. The second and fifth terms of a geometric series are 750 and -6 respectively. Find

(a) the common ratio of the series, (3)

Solution

$$r^3 = \frac{ar^4}{ar} = \frac{-6}{750} = -\frac{1}{125} \Rightarrow \underline{\underline{r = -\tfrac{1}{5}}}.$$

(b) the first term of the series, (2)

Solution

$$a = 750 \div \left(-\tfrac{1}{5}\right) = \underline{\underline{-3750}}.$$

(c) the sum to infinity of the series. (2)

Solution

$$S_\infty = \frac{-3750}{1-(-\frac{1}{5})} = \underline{\underline{-3125}}.$$

6. The second and fourth terms of a geometric series are 7.2 and 5.832 respectively. The common ratio of the series is positive. For this series, find

   (a) the common ratio, (2)

   Solution

   $$r^2 = \frac{5.832}{7.2} = 0.81 \Rightarrow \underline{\underline{r = 0.9}}$$

   because $r > 0$.

   (b) the first term, (2)

   Solution

   $$a = 7.2 \div 0.9 = \underline{\underline{8}}.$$

   (c) the sum of the first 50 terms, giving your answer to 3 decimal places, (2)

   Solution

   $$S_{50} = \frac{8\left[1-(0.9)^{50}\right]}{1-0.9} = 79.587\,697\,98 \text{ (FCD)} = \underline{\underline{79.588 \text{ (3 dp)}}}.$$

   (d) the difference between the sum of infinity and the sum of the first 50 terms, giving your answer to 3 decimal places. (2)

   Solution

   $$S_\infty - S_{50} = 80 - 79.587\,697\,98 \text{ (FCD)} = \underline{\underline{0.412 \text{ (3 dp)}}}.$$

7. The first term of a geometric series is 20 and the common ratio is $\frac{7}{8}$. The sum to infinity of the series is S_∞.

   (a) Find the value of S_∞. (2)

   Solution

   $$S_\infty = \frac{20}{1-\frac{7}{8}} = \underline{\underline{160}}.$$

The sum to N terms of the series is S_N.

(b) Find, to 1 decimal place, the value of S_{12}. (2)

Solution

$$S_{12} = \frac{20\left(1 - \left(\frac{7}{8}\right)^{12}\right)}{1 - \frac{7}{8}} = 127.773\,241\,9 \text{ (FCD)} = \underline{\underline{127.8 \text{ (1 dp)}}}.$$

(c) Find the smallest value of N, for which (4)

$$S_\infty - S_N < 0.5.$$

Solution

$$\begin{aligned} S_\infty - S_N < 0.5 &\Rightarrow 160 - \frac{20\left(1 - \left(\frac{7}{8}\right)^n\right)}{1 - \frac{7}{8}} < 0.5 \\ &\Rightarrow 159.5 < 160 \times \left(1 - \left(\tfrac{7}{8}\right)^n\right) \\ &\Rightarrow \tfrac{319}{320} < 1 - \left(\tfrac{7}{8}\right)^n \\ &\Rightarrow \left(\tfrac{7}{8}\right)^n < \tfrac{1}{320} \\ &\Rightarrow n > \log_{\frac{7}{8}} \tfrac{1}{320} \\ &\Rightarrow n > 43.198 \end{aligned}$$

and so $\underline{\underline{N = 44}}$.

8. A geometric series has first term a, where $a \neq 0$, and common ratio r. The sum to infinity of this series is six times the first term of the series.

(a) Show that $r = \frac{5}{6}$. (2)

Solution

$$6a = \frac{a}{1 - r} \Rightarrow 1 - r = \tfrac{1}{6} \Rightarrow \underline{\underline{r = \tfrac{5}{6}}}.$$

Given that the fourth term of this series 62.5,

(b) find the value of a, (2)

Solution

$$a \times \left(\tfrac{5}{6}\right)^3 = 62.5 \Rightarrow \underline{\underline{a = 108}}.$$

(c) find the difference between sum to infinity and the sum of the first 30 terms, giving your answer to 3 significant figures. (4)

Solution

The sum to infinity is

$$\frac{108}{1 - \frac{5}{6}} = 648.$$

Now,

$$\begin{aligned} S_\infty - S_{30} &= 648 - \frac{108\left[1 - \left(\frac{5}{6}\right)^{30}\right]}{1 - \frac{5}{6}} \\ &= 2.729\,842\,711 \text{ (FCD)} \\ &= \underline{\underline{2.73 \text{ (3 sf)}}}. \end{aligned}$$

9. The third term of a geometric sequence is 324 and the sixth term is 96.

(a) Show that the common ratio of the sequence is $\frac{2}{3}$. (2)

Solution

We have $ar^2 = 324$ and $ar^5 = 96$:

$$r^3 = \frac{96}{324} = \frac{8}{27} \Rightarrow \underline{\underline{r = \frac{2}{3}}}.$$

(b) Find the first term of the sequence. (2)

Solution

$$a = 324 \div \left(\tfrac{2}{3}\right)^2 = \underline{\underline{729}}.$$

(c) Find the sum of the first 15 terms of the sequence. (3)

Solution

$$S_{15} = \frac{729\left[1 - \left(\frac{2}{3}\right)^{15}\right]}{1 - \frac{2}{3}} = 2182.005\,639 \text{ (FCD)} = \underline{\underline{2182 \text{ (nearest integer)}}}.$$

(d) Find the sum to infinity of the sequence. (2)

Solution

$$\frac{729}{1 - \frac{2}{3}} = \underline{\underline{2187}}.$$

10. A car was purchased for £18 000 on 1st January. On 1st January each following year, the value of the car is 80% of its value on 1st January in the previous year.

(a) Show that the value of the car exactly 3 years after it was purchased is £9216. (1)

Solution

$$18\,000 \times 0.8^3 = \underline{\underline{£9216}}.$$

The value of the car falls below £1000 for the first time n years after it was purchased.

(b) Find the value of n. (3)

Solution

$$\begin{aligned} 18\,000 \times 0.8^n < 1000 &\Rightarrow 0.8^n < \tfrac{1}{18} \\ &\Rightarrow n \log 0.8 < \log \tfrac{1}{18} \\ &\Rightarrow n > \frac{\log \frac{1}{18}}{\log 0.8} = 12.952\ldots, \end{aligned}$$

and $\underline{\underline{n = 13}}$.

An insurance company has a scheme to cover the maintenance of the car. The cost is £200 for the first year, and for every following year the cost increases by 12% so that for the 3rd year the cost of the scheme is £250.88.

(c) Find the cost of the scheme for the 5th year, giving your answer to the nearest penny. (2)

Solution

$$200 \times 1.12^4 = 314.703\,872 \text{ (FCD)} = \underline{\underline{\pounds 314.70}}.$$

(d) Find the total cost of the insurance scheme for the 15 years. (3)

Solution

$$\frac{200\left[1 - 1.12^{15}\right]}{1 - 1.12} = 7455.942\,932 \text{ (FCD)} = \underline{\underline{\pounds 7455.94}}.$$

11. A geometric series has first term 5 and common ratio $\frac{4}{5}$. Calculate

(a) the 20th term of the series, to 3 decimal places, (2)

Solution

$$5 \times \left(\tfrac{4}{5}\right)^{19} = 0.072\,057\,594\,04 \text{ (FCD)} = \underline{\underline{0.072 \text{ (3 dp)}}}.$$

(b) the sum to infinity of the series. (2)

Solution

$$\frac{5}{1 - \frac{4}{5}} = \underline{\underline{25}}.$$

Given that the sum to k terms of the series is greater than 24.95,

(c) show that (4)

$$k > \frac{\log 0.002}{\log 0.8},$$

Solution

$$\frac{5\left[1-\left(\frac{4}{5}\right)^k\right]}{1-\frac{4}{5}} > 24.95 \Rightarrow 25\left[1-\left(\tfrac{4}{5}\right)^k\right] > 24.95$$
$$\Rightarrow 1-\left(\tfrac{4}{5}\right)^k > \tfrac{499}{500}$$
$$\Rightarrow \tfrac{1}{500} > \left(\tfrac{4}{5}\right)^k$$
$$\Rightarrow \log\tfrac{1}{500} < k\log\tfrac{4}{5}$$
$$\Rightarrow \underline{\underline{k > \frac{\log 0.002}{\log 0.8}}},$$

as required.

(d) find the smallest possible value of k. (1)

Solution

$$k > \frac{\log 0.002}{\log 0.8} = 27.850\ldots$$

and so we have $\underline{\underline{k = 28}}$.

12. A trading company made a profit of £50 000 in 2006 (Year 1). A model for future trading predicts that the profits will increase year by year in a geometric ratio with common ratio r, $r > 1$. The model therefore predicts that in 2007 (Year 2) a profit of £50 000r will be made.

(a) Write down an expression for predicted profit in Year n. (1)

Solution

$\underline{\underline{\pounds 50\,000 \times r^{n-1}}}$

The model predicts that in Year n, the profit made will exceed £200 000.

(b) Show that (3)

$$n > \frac{\log 4}{\log r} + 1.$$

Solution

$$
\begin{aligned}
50\,000 \times r^{n-1} > 200\,000 &\Rightarrow r^{n-1} > 4 \\
&\Rightarrow (n-1)\log r > \log 4 \\
&\Rightarrow n - 1 > \frac{\log 4}{\log r} \\
&\Rightarrow \underline{\underline{n > \frac{\log 4}{\log r} + 1}}.
\end{aligned}
$$

Using the model with $r = 1.09$,

(c) find the year in which the profit made will first exceed £200 000, (2)

Solution

$$n > \frac{\log 4}{\log 1.09} + 1 = 17.086\,463\,45 \text{ (FCD)}$$

and the answer is $\underline{\underline{\text{18 years or 2023}}}$.

(d) find the total of the profits that will be made by the company over the 10 years from 2006 to 2015 inclusive, giving your answer to the nearest £10 000. (3)

Solution

$$\frac{50\,000\,[1 - 1.09^{10}]}{1 - 1.09} = 759\,646.485\,9 \text{ (FCD)} = \underline{\underline{£760\,000}}.$$

13. (a) All the terms of a given geometric series are positive. The sum of the first two terms is 34 and the sum to infinity is 162. Find

(i) the common ratio, (4)

Solution

Since the sum of the first two terms is 34,

$$a + ar = 34 \Rightarrow a(1 + r) = 34 \Rightarrow a = \frac{34}{1 + r}.$$

Since the sum to inifinity is 162,

$$\frac{a}{1 - r} = 162.$$

Hence

$$\frac{34}{(1 + r)(1 - r)} = 162 \Rightarrow 1 - r^2 = \tfrac{17}{81} \Rightarrow r^2 = \tfrac{64}{81},$$

and hence $\underline{\underline{r = \frac{8}{9}}}$ since $r > 0$.

(ii) the first term. (2)

Solution
Using the previous part,

$$a = \frac{34}{1 + \frac{8}{9}} = \underline{\underline{18}}.$$

(b) A different geometric series has a first term of 42 and a common ratio of $\frac{6}{7}$. Find the smallest value of n for which the sum of the first n terms of the series exceeds 290. (4)

Solution

$$\begin{aligned}\frac{42\left[1 - \left(\frac{6}{7}\right)^n\right]}{1 - \frac{6}{7}} > 290 &\Rightarrow 1 - \left(\frac{6}{7}\right)^n > \tfrac{145}{147} \\ &\Rightarrow \left(\frac{6}{7}\right)^n < \tfrac{2}{147} \\ &\Rightarrow n > \log_{\frac{6}{7}} \tfrac{2}{147} = 27.877\ldots,\end{aligned}$$

and hence $\underline{\underline{28}}$ terms are needed.

Why does the inequality reverse as we move from the penultimate line to the last line? Using the logarithmic base that we did can hide the fact that there is really some division by a negative number going on behind the scenes:

$$\begin{aligned}\left(\tfrac{6}{7}\right)^n < \tfrac{2}{147} &\Rightarrow \log\left(\tfrac{6}{7}\right)^n < \log\tfrac{2}{147} \\ &\Rightarrow n\log\tfrac{6}{7} < \log\tfrac{2}{147} \\ &\Rightarrow n > \frac{\log\frac{2}{147}}{\log\frac{6}{7}} = \log_{\frac{6}{7}}\tfrac{2}{147}.\end{aligned}$$

14. The first three terms of a geometric series are $(k + 4)$, k, and $(2k - 15)$ respectively, where k is a positive constant.

(a) Show that $k^2 - 7k - 60 = 0$. (4)

Solution

$$\frac{k}{k+4} = \frac{2k-15}{k} \Rightarrow (2k-15)(k+4) = k^2$$
$$\Rightarrow 2k^2 - 7k - 60 = k^2$$
$$\Rightarrow \underline{\underline{k^2 - 7k - 60 = 0}},$$

as required.

(b) Hence show that $k = 12$. (2)

Solution

$$k^2 - 7k - 60 = 0 \Rightarrow (k-12)(k+5) = 0 \Rightarrow k = -5 \text{ or } k = 12;$$

since we have a positive constant, $\underline{\underline{k = 12}}$.

(c) Find the common ratio of this series. (2)

Solution

$$r = \frac{12}{12+4} = \frac{12}{16} = \underline{\underline{\frac{3}{4}}}.$$

(d) Find the sum to infinity of this series. (2)

Solution

$$\frac{16}{1-\frac{3}{4}} = \underline{\underline{64}}.$$

15. (a) A geometric series has first term a and common ratio r. Prove that the sum of the first n terms of this series is given by (4)

$$\frac{a(1-r^n)}{1-r}.$$

Solution

Let S_n be the first n terms. So,

$$\begin{aligned} S_n &= a + ar + ar^2 + \ldots + ar^{n-1} \\ rS_n &= ar + ar^2 + ar^3 + \ldots + ar^n \\ \Rightarrow S_n - rS_n &= a - ar^n \\ \Rightarrow (1-r)S_n &= a(1-r^n) \\ \Rightarrow S_n &= \underline{\underline{\frac{a(1-r^n)}{1-r}}}. \end{aligned}$$

Mr King will be paid a salary of £35 000 in the year 2005. Mr King's contract promises a 4% increase in salary every year, the first increase being given in 2006, so that his annual salaries form a geometric sequence.

(b) Find, to the nearest £100, the Mr King's salary in the year 2008. (2)

Solution

$$£35\,000 \times 1.04^3 = £39\,373.24 = \underline{\underline{£39\,400}}.$$

Mr King will receive a salary each year from 2005 until he retires at the end of 2024.

(c) Find, to the nearest £1000, the total amount of salary he will receive in the period from 2005 until he retires at the end of 2024. (4)

Solution

$a = 35\,000$, $r = 1.04$, and $n = 20$:

$$\frac{£35\,000\left[1-(1.04)^{20}\right]}{1-1.04} = £1\,042\,232.75 \text{ (FCD)} = \underline{\underline{1\,042\,000 \text{ (nearest £1000)}}}.$$

16. The second and third terms of a geometric series are 192 and 144 respectively. For this series, find

(a) the common ratio, (2)

Solution

$r = \frac{144}{192} = \underline{\underline{\frac{3}{4}}}$.

(b) the first term, (2)

Solution
$a = 192 \div \frac{3}{4} = \underline{\underline{256}}$.

(c) the sum to infinity, (2)

Solution

$$S_\infty = \frac{256}{1 - \frac{3}{4}} = \underline{\underline{1024}}.$$

(d) the smallest value of n for which the sum of the first n terms of the series exceeds 1000. (4)

Solution

$$\begin{aligned} \frac{256\left[1 - \left(\frac{3}{4}\right)^n\right]}{1 - \frac{3}{4}} > 1000 \Rightarrow 1024\left[1 - \left(\frac{3}{4}\right)^n\right] > 1000 \\ \Rightarrow 1 - \left(\tfrac{3}{4}\right)^n > \tfrac{125}{128} \\ \Rightarrow \tfrac{3}{128} > \left(\tfrac{3}{4}\right)^n \\ \Rightarrow n > \log_{\frac{3}{4}} \tfrac{3}{128} \\ \Rightarrow n > 13.047\ldots, \end{aligned}$$

and so the correct answer is $\underline{\underline{\text{14 terms}}}$.

17. The adult population of a town at 25 000 at the end of Year 1. A model predicts that the adult population of the town will increase by 3% each year, forming a geometric sequence.

(a) Show the predicted adult population at the end of Year 2 is 25 750. (1)

Solution
$25\,000 \times 1.03 = \underline{\underline{25\,750}}$.

(b) Write down the common ratio of the geometric sequence. (1)

Solution
$\underline{\underline{1.03}}$.

The model predicts that Year N will be the first year in which the adult population of the town exceeds 40 000.

(c) Show that (3)

$$(N-1)\log 1.03 > \log 1.6.$$

Solution

$$\begin{aligned} 25\,000 \times 1.03^{N-1} > 40\,000 &\Rightarrow 1.03^{N-1} > 1.6 \\ &\Rightarrow \log 1.03^{N-1} > \log 1.6 \\ &\Rightarrow \underline{\underline{(N-1)\log 1.03 > \log 1.6}}. \end{aligned}$$

(d) Find the value of N. (2)

Solution

$$N - 1 > \frac{\log 1.6}{\log 1.03} = 15.900\ldots \Rightarrow N = 16.900\ldots$$

so $\underline{\underline{N = 17}}$.

At the end of each year, each member of the adult population of the town will give £1 to a charity fund. Assuming the population model,

(e) find the total amount that will be given to the charity fund for the 10 years from the end of Year 1 to the end of Year 10, giving your answer to the nearest £1000. (3)

Solution

$$\frac{25\,000\,[1 - 1.03^{10}]}{1 - 1.03} = 286\,596.982\,8 \text{ (FCD)} = \underline{\underline{\pounds 287\,000}}.$$

18. The first terms terms of a geometric series are $4p$, $(3p + 15)$, and $(5p + 20)$ respectively, where p is a positive constant.

(a) Show that $11p^2 - 10p - 225 = 0$. (4)

Solution

$$\frac{3p+15}{4p} = \frac{5p+20}{3p+15} \Rightarrow 4p(5p+20) = (3p+15)^2$$
$$\Rightarrow 20p^2 + 80p = 9p^2 + 90p + 225$$
$$\Rightarrow \underline{\underline{11p^2 - 10p - 225 = 0}}.$$

(b) Hence show that $p = 5$. (2)

Solution

$$(11p+45)(p-5) = 0 \Rightarrow p = 5 \text{ or } p = -4\tfrac{1}{11}.$$

Hence, because the terms are all positive, $\underline{\underline{p = 5}}$.

(c) Find the common ratio of this series. (2)

Solution

$r = \frac{30}{20} = \underline{\underline{\frac{3}{2}}}$.

(d) Find the sum of the ten terms of the series, giving your answer to the nearest integer. (3)

Solution

$a = 20, r = \frac{3}{2}, n = 10$:

$$\frac{20\left[1 - \left(\frac{3}{2}\right)^{10}\right]}{1 - \frac{3}{2}} = 2266.601\,563 \text{ (FCD)} = \underline{\underline{2267 \text{ (nearest integer)}}}.$$

19. A geometric series is $a + ar + ar^2 + \ldots$.

(a) Prove that the sum of the first n terms of this series is given by (4)

$$S_n = \frac{a(1-r^n)}{1-r}.$$

Solution

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \ldots + ar^{n-1} \\
rS_n &= ar + ar^2 + ar^3 + \ldots + ar^n \\
\Rightarrow S_n - rS_n &= a - ar^n \\
\Rightarrow (1-r)S_n &= a(1-r^n) \\
\Rightarrow \underline{\underline{S_n = \frac{a(1-r^n)}{1-r}}}.
\end{aligned}
$$

(b) Find (3)

$$\sum_{k=1}^{10} 100(2^k).$$

Solution

We use $a = 200$, $r = 10$, and $n = 10$:

$$\frac{100\left[1-2^{10}\right]}{1-2} = \underline{\underline{204\,600}}.$$

(c) Find the sum to infinity of the geometric series (3)

$$\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \ldots.$$

Solution

$$
\begin{aligned}
\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \ldots &= \frac{5}{6}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right] \\
&= \frac{5}{6} \times \frac{1}{1-\frac{1}{3}} \\
&= \underline{\underline{1\tfrac{1}{4}}}.
\end{aligned}
$$

(d) State the condition for an infinite geometric series with common ratio r to be convergent. (1)

Solution

$\underline{\underline{|r| < 1}}$.

20. A geometric series has first term a and common ratio r. The second terms of the series is 4 and the sum to infinity of the series is 25.

(a) Show that $25r^2 - 25r + 4 = 0$. (4)

Solution

$$\begin{aligned} 25 = \frac{a}{1-r} &\Rightarrow 25(1-r) = a \\ &\Rightarrow 25r(1-r) = ar \\ &\Rightarrow 25r - 25r^2 = 4 \\ &\Rightarrow \underline{\underline{0 = 25r^2 - 25r + 4}}. \end{aligned}$$

(b) Find two possible values of r. (2)

Solution

$$25r^2 - 25r + 4 = 0 \Rightarrow (5r-1)(5r-4) = 0 \Rightarrow \underline{\underline{r = \tfrac{1}{5}}} \text{ or } \underline{\underline{r = \tfrac{4}{5}}}.$$

(c) Find the corresponding two possible values of a. (2)

Solution

$a = 4 \div \frac{1}{5} = \underline{\underline{20}}$ or $a = 4 \div \frac{4}{5} = \underline{\underline{5}}$.

(d) Show the the sum, S_n, of the first n terms of the series is given by (1)

$$S_n = 25(1 - r^n).$$

Solution

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \times (1 - r^n) = \underline{\underline{25(1-r^n)}}.$$

Given that r takes the larger of its two possible values,

(e) find the smallest value of n for which S_n exceeds 24. (2)

Solution

$$\begin{aligned} 25\left[1 - \left(\tfrac{4}{5}\right)^n\right] > 24 &\Rightarrow 1 - \left(\tfrac{4}{5}\right)^n > \tfrac{24}{25} \\ &\Rightarrow \tfrac{1}{25} > \left(\tfrac{4}{5}\right)^n \\ &\Rightarrow \log \tfrac{1}{25} < \log \left(\tfrac{4}{5}\right)^n \\ &\Rightarrow \log \tfrac{1}{25} < n \log \tfrac{4}{5} \\ &\Rightarrow \frac{\log \frac{1}{25}}{\log \frac{4}{5}} < n \\ &\Rightarrow 14.425\ldots < n, \end{aligned}$$

and hence the solution is $\underline{\underline{n = 15}}$.

21. The first term of a geometric series is 120. The sum of infinity of the series is 480.

(a) Show the common ratio, r, is $\frac{3}{5}$. (3)

Solution

$$\frac{120}{1-r} = 480 \Rightarrow \tfrac{1}{4} = 1 - r \Rightarrow \underline{\underline{r = \tfrac{3}{4}}}.$$

(b) Find, to 2 decimal places, the difference between the 5th and 6th terms. (2)

Solution

$$120\left(\tfrac{3}{4}\right)^4 - 120\left(\tfrac{3}{4}\right)^5 = 9\tfrac{63}{128} = \underline{\underline{9.49 \text{ (2 dp)}}}.$$

(c) Calculate the sum of the first 7 terms. (2)

Solution

$$\frac{120\left[1 - \left(\frac{3}{4}\right)^7\right]}{1 - \frac{3}{4}} = 415.927\,734\,4 \text{ (FCD)} = \underline{\underline{416 \text{ (nearest integer)}}}.$$

The sum of the first n terms of the series is greater than 300.

(d) Calculate the smallest possible value of n. (4)

Solution

$$
\begin{aligned}
\frac{120\left[1-\left(\frac{3}{4}\right)^n\right]}{1-\frac{3}{4}} > 300 &\Rightarrow 1-\left(\tfrac{3}{4}\right)^n > \tfrac{5}{8} \\
&\Rightarrow \tfrac{3}{8} > \left(\tfrac{3}{4}\right)^n \\
&\Rightarrow \log\tfrac{3}{8} > \log\left(\tfrac{3}{4}\right)^n \\
&\Rightarrow \log\tfrac{3}{8} > n\log\tfrac{3}{4} \\
&\Rightarrow \frac{\log\frac{3}{8}}{\log\frac{3}{4}} < n \\
&\Rightarrow 3.409\ldots < n,
\end{aligned}
$$

and hence the answer is $\underline{\underline{n=4}}$.

22. The first three terms of a geometric sequence are

$$7k-5,\ 5k-7,\ \text{and}\ 2k+10,$$

where k is a constant.

(a) Show that $11k^2 - 130k + 99 = 0$. (4)

Solution

$$
\begin{aligned}
\frac{5k-7}{7k-5} = \frac{2k+10}{5k-7} &\Rightarrow (5k-7)^2 = (7k-5)(2k+10) \\
&\Rightarrow 25k^2 - 70k + 49 = 14k^2 + 60k - 50 \\
&\Rightarrow \underline{\underline{11k^2 - 130k + 99 = 0}},
\end{aligned}
$$

as required.

Given that k is not an integer,

(b) show that $k = \frac{9}{11}$. (2)

Solution

$$
\begin{aligned}
11k^2 - 130k + 99 = 0 &\Rightarrow (11k-9)(k-11) = 0 \\
&\Rightarrow 11k-9 = 0 \text{ or } k-11 = 0 \\
&\Rightarrow \underline{\underline{k = \tfrac{9}{11}}}
\end{aligned}
$$

because $k = 11$ is an integer.

For this value of k,

(c) (i) evaluate the fourth term of the sequence, giving your answer as an exact fraction, (6)

Solution
1st term is $\frac{8}{11}$, the 2nd term is $-\frac{32}{11}$, and the 3rd term is $\frac{128}{11}$. So $r = -4$ and we have the 4th term
$$\tfrac{8}{11} \times (-4)^3 = \underline{\underline{-\tfrac{512}{11}}}.$$

(ii) evaluate the sum of the first ten terms of the sequence.

Solution
$a = \frac{8}{11}, r = -4, n = 10$:
$$\frac{\frac{8}{11}\left[1 - (-4)^{10}\right]}{1 - (-4)} = -\underline{\underline{152\,520}}.$$