Further Pure Mathematics 3: Part 2

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Further Mathematics

Vector Product

- Vector product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector in the direction given by the right-hand rule.
- If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then the vector product can be calculated using the symbolic determinant
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$
- The area of the triangle OAB is given by $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B respectively.
- The area of the triangle ABC is given by
$$\frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$
where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the position vectors of A, B, and C respectively.
- The volume of a parallelipiped is given by the modulus of the *triple scalar product*
$$|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$$
- The volume of a tetrahedron is given by
$$\frac{1}{6}|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$$

Lines

- The cartesian equation of a line in three dimensions has the form
$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}.$$
- A vector equation of a line, passing through a point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$, may be written as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, or $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$.
- The angle θ between the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and the plane $\mathbf{r} \cdot \mathbf{n} = p$ is given by
$$\sin\theta = \left|\frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}|\,|\mathbf{n}|}\right|$$
- The shortest distance between the skew lines $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + s\mathbf{d}$ is
$$\left|\frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|}\right|$$

Planes

- The cartesian equation of the plane is
$$ax + by + cz = d.$$
- A vector equation of the plane is $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a normal vector to the plane.
- The distance between the plane $\mathbf{r} \cdot \mathbf{n} = p$ and the origin is
$$\frac{|p|}{|\mathbf{n}|}$$
- The angle θ between the planes $\mathbf{r} \cdot \mathbf{n}_1 = p_1$ and the plane $\mathbf{r} \cdot \mathbf{n}_2 = p_2$ is given by
$$\cos\theta = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right|$$

Hyperbolic Integrals

$$\int \sinh x \,\mathrm{d}x = \cosh x + c$$
$$\int \cosh x \,\mathrm{d}x = \sinh x + c$$
$$\int \tanh x \,\mathrm{d}x = \ln(\cosh x) + c$$
$$\int \operatorname{cosech} x \,\mathrm{d}x = \ln\left|\tanh \tfrac{1}{2}x\right| + c$$
$$\int \operatorname{sech} x \,\mathrm{d}x = 2\arctan\left(e^x\right) + c$$
$$\int \coth x \,\mathrm{d}x = \ln|\sinh x| + c$$

Standard Integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}} \,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c,\ |x| < a$$
$$\int \frac{1}{\sqrt{x^2 - a^2}} \,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + c,\ x > a$$
$$\int \frac{1}{\sqrt{a^2 + x^2}} \,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) + c$$
$$\int \frac{1}{a^2 + x^2} \,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$
$$\int \frac{1}{a^2 - x^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + c$$
$$\int \frac{1}{x^2 - a^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + c$$

You should also be able to find the integral of the inverse functions by using integration by parts in the form $1 \times$ inverse function.

Arc Length

There are three methods of calculating arc length:
$$\int_{x_A}^{x_B} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \,\mathrm{d}x$$
$$= \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \,\mathrm{d}y$$
$$= \int_{t_A}^{t_B} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

Surface Area

There are five methods of calculating the surface area of revolution:

around x-axis $= 2\pi \int_{x_A}^{x_B} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \,\mathrm{d}x$
$$= 2\pi \int_{t_A}^{t_B} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

around y-axis $= 2\pi \int_{y_A}^{y_B} x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \,\mathrm{d}y$
$$= 2\pi \int_{x_A}^{x_B} x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \,\mathrm{d}x$$
$$= 2\pi \int_{t_A}^{t_B} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t$$

Reduction formulae

You need to be able to work with reduction formulae in a wide variety of contexts (powers of trigonometric and hyperbolic functions, definite and indefinite integrals) and you should ensure that you are able to quickly identify the appropriate technique (such as integration by parts) for establishing the required result.

Matrices

- A transformation is linear if and only if it can be represented by a matrix.
- If $\mathbf{A}$ is the matrix of a transformation then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Inverse of a 3×3 Matrix

To form the inverse of a non-singular matrix,
- form the matrix of minors, i.e., replace each entry with the determinant of the 2 × 2 matrix that is left when that row and column are knocked out,
- form the matrix of cofactors, i.e., either leave each entry alone of change its sign according to the alternating pattern starting in the top left-hand corner,
- form the transpose of the matrix of cofactors,
- multiply by the reciprocal of the determinant.

Eigenvalues and eigenvectors

- If λ is a scalar and $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ then λ is an *eigenvalue* and $\mathbf{x}$ is an *eigenvector*.
- The eigenvalues λ are found by solving the *characteristic equation*, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
- The sum of the eigenvalues is equal to the trace of the matrix, i.e., the sum of the entries on its leading diagonal.
- The product of the eigenvalues is equal to the determinant of the matrix.
- Every matrix satisfies it characteristic equation; this is known as the *Cayley-Hamilton theorem*.

Diagonalisation of a matrix

In order to diagonalise a symmetric matrix,
- find the eigenvalues and the eigenvectors,
- normalise the eigenvectors,
- form a matrix $\mathbf{P}$ whose columns are the normalised eigenvectors,
- note that the matrix $\mathbf{P}$ is orthogonal, i.e., $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$; this is because eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are themselves orthogonal,
- then $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix whose entries are the eigenvalues in the same order as the order of the eigenvectors in the matrix $\mathbf{P}$.