Dr Oliver Mathematics
Area by Determinant 2

In this note, we will investigate the area by determinant.

In Cambridge Additional Mathematics November 2015 Paper 2 Version 2, we were asked to find the area of triangle ABE, where $A(-3, 2)$, $B(9, 8)$, and $E(0, 11)$.

I will present the answer:

$$\begin{aligned}
\text{Area of } ABE &= \tfrac{1}{2} \left| \begin{matrix} -3 & 9 & 0 & -3 \\ 2 & 8 & 11 & 2 \end{matrix} \right| \\
&= \tfrac{1}{2} \left| -24 + 99 - 18 + 33 \right| \\
&= \tfrac{1}{2} \left| 90 \right| \\
&= 45.
\end{aligned}$$

What?!

In other papers, the 'solution' was "complete method for entire area" or "method for area."

Now, as far as I am aware, you can only do the determinants for square matrices. But I was intrigued: we have A, B, E, and A as columns and I know that $(-3) \times 8 = -24$, $9 \times 11 = 99$, etc.

Then I realised: split in to three 2×2 determinants, evaluate each of them, then add the three values together. In this case,

$$\left| \begin{matrix} -3 & 9 & 0 & -3 \\ 2 & 8 & 11 & 2 \end{matrix} \right| + \left| \begin{matrix} -3 & 9 & 0 & -3 \\ 2 & 8 & 11 & 2 \end{matrix} \right| + \left| \begin{matrix} -3 & 9 & 0 & -3 \\ 2 & 8 & 11 & 2 \end{matrix} \right|$$

$$\begin{aligned}
&= [(-3 \times 8) - (2 \times 9)] + [(9 \times 11) - (8 \times 0)] + [(0 \times 2) - (11 \times (-3))] \\
&= -42 + 99 + 33 \\
&= 90.
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{Area of } ABE &= \tfrac{1}{2} \left| \begin{matrix} -3 & 9 & 0 & -3 \\ 2 & 8 & 11 & 2 \end{matrix} \right| \\
&= \tfrac{1}{2} \left| 90 \right| \\
&= 45.
\end{aligned}$$

1 Proof

$$\begin{aligned}
\text{Area}_1 &= \frac{1}{2}\left|\det\begin{pmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{pmatrix}\right| \\
&= \frac{1}{2}\left|a(d-f) - b(c-e) + 1(cf-de)\right| \\
&= \frac{1}{2}\left|ad - af - bc + be + cf - de\right| \\
&= \frac{1}{2}\left|ad + be + cf - af - bc - de\right| \\
&= \frac{1}{2}\left|(ad-bc) + (cf-de) + (be-af)\right|
\end{aligned}$$

and

$$\begin{aligned}
\text{Area}_2 &= \frac{1}{2}\begin{vmatrix} a & c & e & a \\ b & d & f & b \end{vmatrix} \\
&= \frac{1}{2}\left|\begin{vmatrix} a & c \\ b & d \end{vmatrix} + \begin{vmatrix} c & e \\ d & f \end{vmatrix} + \begin{vmatrix} e & a \\ f & b \end{vmatrix}\right| \\
&= \frac{1}{2}\left|(ad-bc) + (cf-de) + (be-af)\right|
\end{aligned}$$

As $\text{Area}_1 = \text{Area}_2$, we have proved the claim.

2 A Second Example

In the same paper, we want to find the area of triangle CDE, where $C(0, 3.5)$, $D(3, 5)$, and $E(0, 11)$. Now,

$$\begin{aligned}
\text{area of } CDE &= \frac{1}{2}\left|\begin{vmatrix} 0 & 3 \\ 3.5 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 5 & 11 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 11 & 3.5 \end{vmatrix}\right| \\
&= \frac{1}{2}\left|(0-10.5) + (33-0) + (0-0)\right| \\
&= \frac{1}{2}\left|22.5\right| \\
&= 11.25,
\end{aligned}$$

which is the correct answer.

3 We Can Extend The Formula

Example 1 (2012 November Paper 1 Version 2 Q7)
The vertices of the trapezium $ABCD$ are the points $A(-5, 4)$, $B(8, 4)$, $C(6, 8)$, and $D(3, 8)$.

Find the area of the trapezium $ABCD$.

Solution 1

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}\begin{vmatrix} -5 & 8 & 6 & 3 & -5 \\ 4 & 4 & 8 & 8 & 4 \end{vmatrix} \\
&= \tfrac{1}{2}\left| \begin{vmatrix} -5 & 8 \\ 4 & 4 \end{vmatrix} + \begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 6 & 3 \\ 8 & 8 \end{vmatrix} + \begin{vmatrix} 3 & -5 \\ 8 & 4 \end{vmatrix} \right| \\
&= \tfrac{1}{2}\left|(-20-32)+(64-24)+(48-24)+(12+40)\right| \\
&= \tfrac{1}{2}\left|-52+40+24+52\right| \\
&= \tfrac{1}{2}\left|64\right| \\
&= \underline{\underline{32}}.
\end{aligned}$$

Example 3 (2012 November Paper 2 Version 3 Q8)
The vertices of the quadrilateral $ABCD$ are the points $A(-4, 6)$, $B(6, -4)$, $C(10, 4)$, and $D(8, 10)$.

Find the area of the quadrilateral $ABCD$.

Solution 2

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}\begin{vmatrix} -4 & 6 & 10 & 8 & -4 \\ 6 & -4 & 4 & 10 & 6 \end{vmatrix} \\
&= \tfrac{1}{2}\left| \begin{vmatrix} -4 & 6 \\ 6 & -4 \end{vmatrix} + \begin{vmatrix} 6 & 10 \\ -4 & 4 \end{vmatrix} + \begin{vmatrix} 10 & 8 \\ 4 & 10 \end{vmatrix} + \begin{vmatrix} 8 & -4 \\ 10 & 6 \end{vmatrix} \right| \\
&= \tfrac{1}{2}\left|(16-36)+(24+40)+(100-32)+(48+40)\right| \\
&= \tfrac{1}{2}\left|-20+64+68+88\right| \\
&= \tfrac{1}{2}\left|200\right| \\
&= \underline{\underline{100}}.
\end{aligned}$$