Dr Oliver Mathematics
Matrix Multiplication: Some Basic Transformations

In this note, we will look at some basic transformations.

Recall,

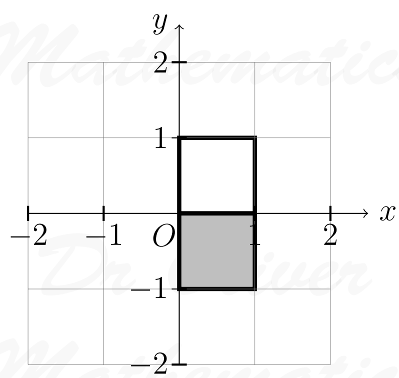
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

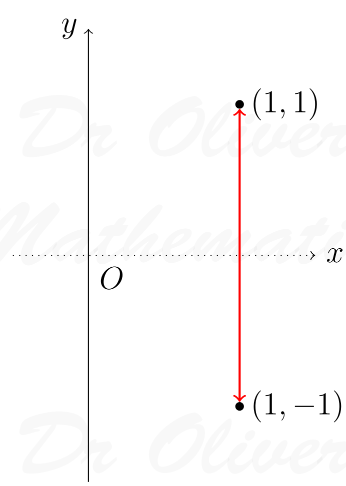
and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

1 Reflection in the x-axis

The image of the unit square is





$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\-1\end{pmatrix}$$

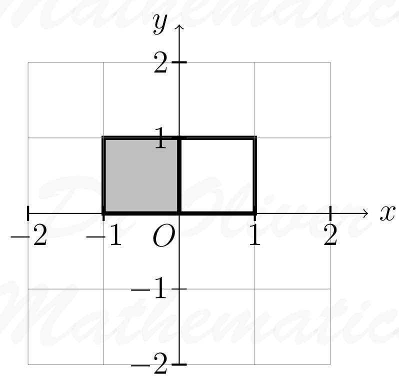
so the required matrix is

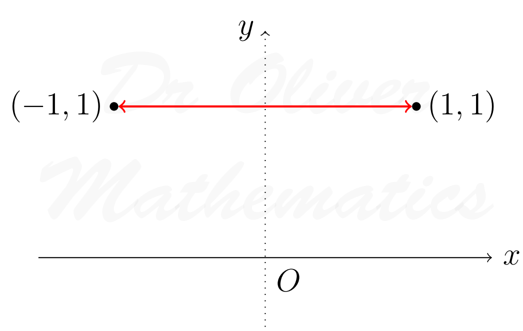
$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

$$\text{Matrix of the transformation}: \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$
$$\text{Inverse of the matrix}: \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$
$$\text{Determinant of the matrix}: -1$$

2 Reflection in the y-axis

The image of the unit square is





$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

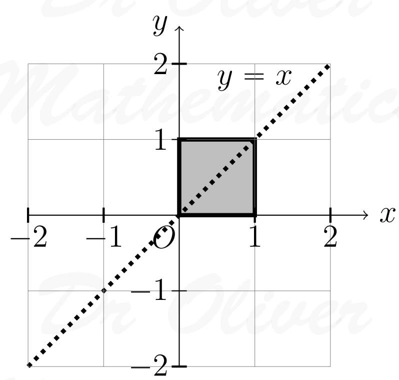
$$\text{Matrix of the transformation} : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

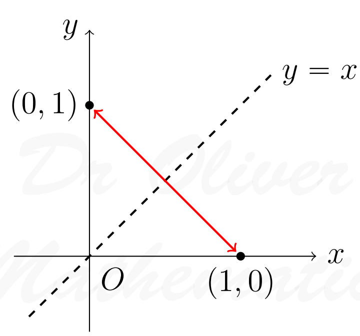
$$\text{Inverse of the matrix} : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Determinant of the matrix} : -1$$

3 Reflection in the line $y = x$

The image of the unit square is





$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

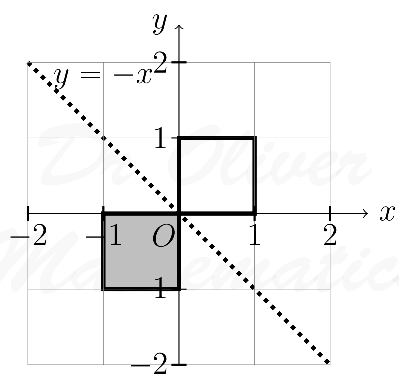
$$\text{Matrix of the transformation} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

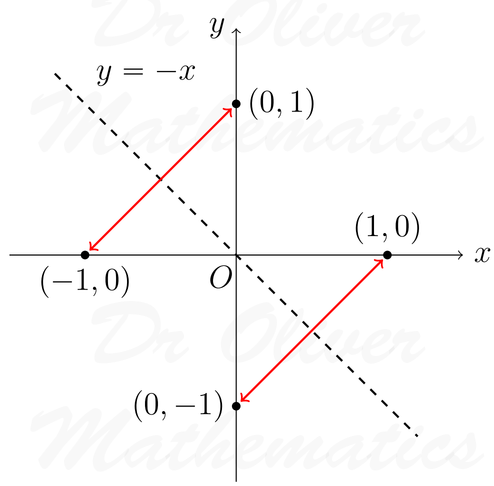
$$\text{Inverse of the matrix} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Determinant of the matrix} : -1$$

4 Reflection in the line $y = -x$

The image of the unit square is





$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

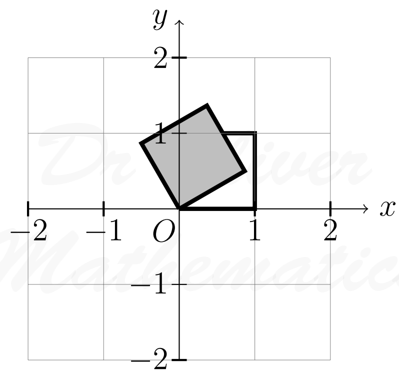
$$\text{Matrix of the transformation : } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

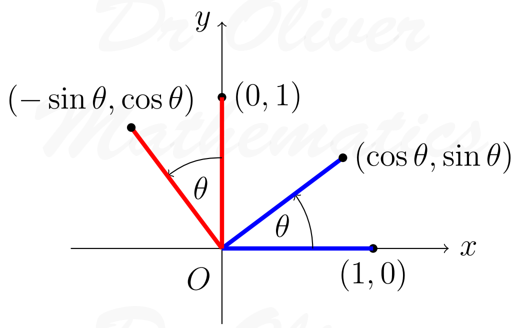
$$\text{Inverse of the matrix : } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\text{Determinant of the matrix : } -1$$

5 Rotation, about O, through θ° anti-clockwise

The image of the unit square is





$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

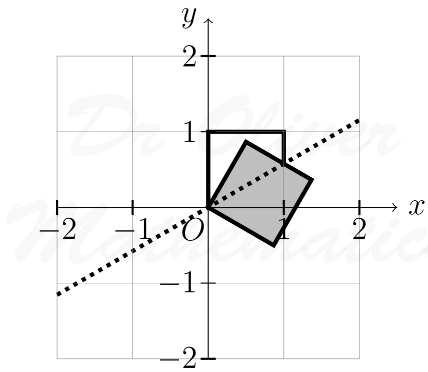
$$\text{Matrix of the transformation : } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\text{Inverse of the matrix : } \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\text{Determinant of the matrix : } +1$$

6 Reflection in the line $y = (\tan\theta^\circ)x$

The image of the unit square is



How did we do this? Well, we do

(a) rotation, about O, through θ° *clockwise*,

(b) reflection in the x-axis, and

(c) rotation, about O, through θ° *anti-clockwise*.

$$\begin{aligned}
& \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
= & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\
= & \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} \\
= & \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -(\cos^2\theta - \sin^2\theta) \end{pmatrix} \\
= & \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.
\end{aligned}$$

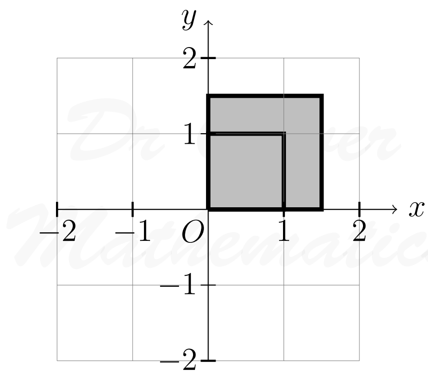
$$\text{Matrix of the transformation} : \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\text{Inverse of the matrix} : \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\text{Determinant of the matrix} : -1$$

7 Enlargement, centre O, scale factor k

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}k\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\k\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix}k & 0\\0 & k\end{pmatrix}.$$

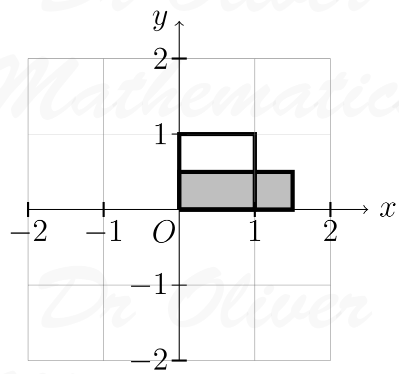
$$\text{Matrix of the transformation}: \begin{pmatrix}k & 0\\0 & k\end{pmatrix}$$

$$\text{Inverse of the matrix}: \begin{pmatrix}\frac{1}{k} & 0\\0 & \frac{1}{k}\end{pmatrix}$$

$$\text{Determinant of the matrix}: k^2$$

8 Two-way stretch, factors a and b, in the direction of the x- and y-axes, respectively

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}a\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\b\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix}a & 0\\0 & b\end{pmatrix}.$$

$$\text{Matrix of the transformation}: \begin{pmatrix}a & 0\\0 & b\end{pmatrix}$$

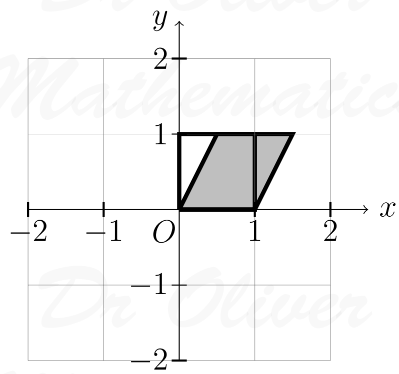
$$\text{Inverse of the matrix}: \begin{pmatrix}\frac{1}{a} & 0\\0 & \frac{1}{b}\end{pmatrix}$$

$$\text{Determinant of the matrix}: ab$$

A *shear* is a transformation in which all points along a given line L remain fixed while other points are shifted parallel to L by a distance proportional to their perpendicular distance from L. Note: shearing a plane figure does not change its area.

9 Shear with x-axis invariant, factor k

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}k\\1\end{pmatrix}$$

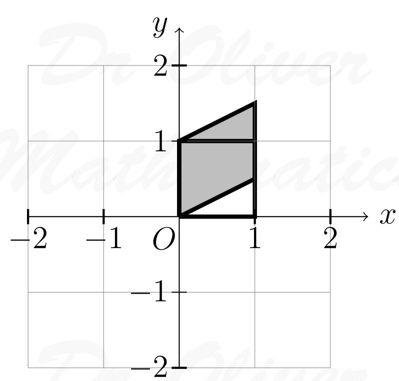
so the required matrix is

$$\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}.$$

$$\text{Matrix of the transformation : } \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$
$$\text{Inverse of the matrix : } \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$
$$\text{Determinant of the matrix : } +1$$

10 Shear with y-axis invariant, factor k

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\k\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix}1 & 0\\k & 1\end{pmatrix}.$$

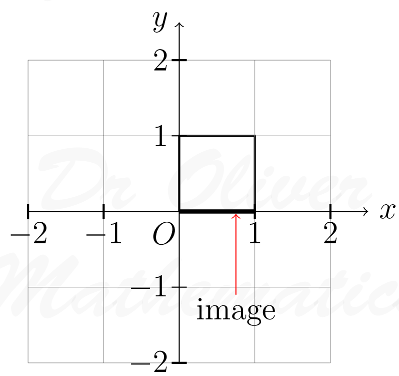
$$\text{Matrix of the transformation : } \begin{pmatrix}1 & 0\\k & 1\end{pmatrix}$$

$$\text{Inverse of the matrix : } \begin{pmatrix}1 & 0\\-k & 1\end{pmatrix}$$

$$\text{Determinant of the matrix : } +1$$

11 Projection, parallel to the y-axis, onto the x-axis

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}.$$

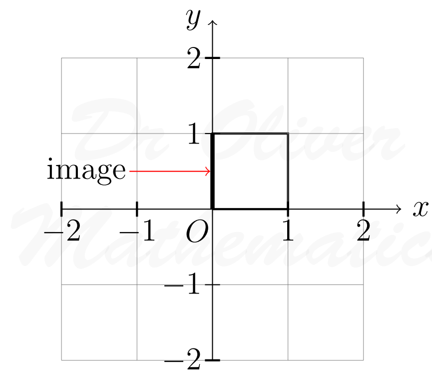
$$\text{Matrix of the transformation : } \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

Inverse of the matrix : does not exist

Determinant of the matrix : 0

12 Projection, parallel to the x-axis, onto the y-axis

The image of the unit square is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$

so the required matrix is

$$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}.$$

$$\text{Matrix of the transformation : } \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$
$$\text{Inverse of the matrix : does not exist}$$
$$\text{Determinant of the matrix : } 0$$