Dr Oliver Mathematics
Applied Mathematics: Differentiation

The total number of marks available is 49.
You must write down all the stages in your working.

1. Differentiate, and simplify as appropriate,

   (a) $\mathrm{f}(x) = \exp(\tan \frac{1}{2}x)$, where $-\pi < x < \pi$, (3)

   Solution

   $$\begin{aligned}\mathrm{f}(x) = \exp(\tan \tfrac{1}{2}x) &\Rightarrow \mathrm{f}'(x) = \exp(\tan \tfrac{1}{2}x) \cdot \sec^2 \tfrac{1}{2}x \cdot \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{\mathrm{f}'(x) = \tfrac{1}{2}\sec^2 \tfrac{1}{2}x \, \exp(\tan \tfrac{1}{2}x)}}.\end{aligned}$$

   (b) $\mathrm{g}(x) = (x^3 + 1)\ln(x^3 + 1)$, where $x > 0$. (3)

   Solution

   $$u = x^3 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$$
   $$v = \ln(x^3 + 1) \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3x^2}{x^3 + 1}$$

   $$\begin{aligned}\mathrm{g}(x) = (x^3 + 1)\ln(x^3 + 1) &\Rightarrow \mathrm{g}'(x) = 3x^2 \cdot \ln(x^3 + 1) + (x^3 + 1) \cdot \frac{3x^2}{(x^3 + 1)} \\ &\Rightarrow \underline{\underline{\mathrm{g}'(x) = 3x^2 \, \ln(x^3 + 1) + 3x^2}}.\end{aligned}$$

2. Given that (5)

   $$y = \ln(1 + \sin x),$$

   where $0 < x < \pi$, show that

   $$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{-1}{1 + \sin x}.$$

Solution

$$y = \ln(1 + \sin x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{1 + \sin x}$$

$$u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$
$$v = 1 + \sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(1 + \sin x)(-\sin x) - \cos x(\cos x)}{(1 + \sin x)^2}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-\sin x - 1}{(1 + \sin x)^2}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-1}{1 + \sin x}}},$$

as required.

3. (a) Given (2)

$$\mathrm{f}(x) = x \tan 2x$$

for $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$, obtain an expression for $\mathrm{f}'(x)$.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = \tan 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\sec^2 2x$$

and

$$\begin{aligned} \mathrm{f}(x) = x\tan 2x &\Rightarrow \mathrm{f}'(x) = (x)(2\sec^2 2x) + (1)(\tan 2x) \\ &\Rightarrow \underline{\underline{\mathrm{f}'(x) = 2x\sec^2 2x + \tan 2x}}. \end{aligned}$$

(b) Show that (3)

$$\mathrm{f}''(x) = 4\sec^2 2x(1 + 2x\tan 2x).$$

Solution

$$\begin{aligned} u = 2x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \\ v = \sec^2 2x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = (2\sec 2x)(\sec 2x\tan 2x)(2) = 4\sec^2 2x\tan 2x \end{aligned}$$

and

$$\begin{aligned} & \mathrm{f}'(x) = 2x\sec^2 2x + \tan 2x \\ \Rightarrow\quad & \mathrm{f}''(x) = (2x)(4\sec^2 2x\tan 2x) + (2)(\sec^2 2x) + 2\sec^2 2x \\ \Rightarrow\quad & \mathrm{f}''(x) = 8x\sec^2 2x\tan 2x + 4\sec^2 2x \\ \Rightarrow\quad & \underline{\underline{\mathrm{f}''(x) = 4\sec^2 2x(1 + 2x\tan 2x)}}, \end{aligned}$$

as required.

(c) Hence find the exact value of (4)

$$\int_0^{\frac{1}{6}\pi} \frac{1 + 2x\tan 2x}{\cos^2 2x}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
\int_0^{\frac{1}{6}\pi} \frac{1 + 2x \tan 2x}{\cos^2 2x} \, \mathrm{d}x &= \int_0^{\frac{1}{6}\pi} \sec^2 2x (1 + 2x \tan 2x) \, \mathrm{d}x \\
&= \tfrac{1}{4} \int_0^{\frac{1}{6}\pi} 4 \sec^2 2x (1 + 2x \tan 2x) \, \mathrm{d}x \\
&= \tfrac{1}{4} \left[2x \sec^2 2x + \tan 2x \right]_{x=0}^{\frac{1}{6}\pi} \\
&= \tfrac{1}{4} \left[\left(\tfrac{4}{3}\pi + \sqrt{3} \right) - (0 + 0) \right] \\
&= \underline{\underline{\tfrac{1}{12} \left(4\pi + 3\sqrt{3} \right)}}.
\end{aligned}
$$

4. Differentiate the following, simplifying your answers as appropriate.

(a) $\mathrm{f}(x) = \mathrm{e}^{2x} \tan x$, $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. (3)

Solution

$$
\begin{aligned}
u = \mathrm{e}^{2x} &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x} \\
v = \tan x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{f}(x) = \mathrm{e}^{2x} \tan x &\Rightarrow \mathrm{f}'(x) = (\mathrm{e}^{2x})(\sec^2 x) + (2\mathrm{e}^{2x})(\tan x) \\
&\Rightarrow \underline{\underline{\mathrm{f}'(x) = \mathrm{e}^{2x}(\sec^2 x + 2 \tan x)}}.
\end{aligned}
$$

(b) $\mathrm{g}(x) = \dfrac{\cos 2x}{x^3}$. (4)

Solution

$$
\begin{aligned}
u = \cos 2x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2 \sin 2x \\
v = x^3 &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 3x^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{g}(x) = \frac{\cos 2x}{x^3} \Rightarrow \mathrm{g}'(x) &= \frac{(x^3)(-2\sin 2x) - (\cos 2x)(3x^2)}{(x^3)^2} \\
\Rightarrow \mathrm{g}'(x) &= \frac{x^2(-3\cos 2x - 2x\sin 2x)}{x^6} \\
\Rightarrow \mathrm{g}'(x) &= \underline{\underline{\frac{-3\cos 2x - 2x\sin 2x}{x^4}}}.
\end{aligned}
$$

5. Differentiate the following, simplifying where possible.

(a) $\mathrm{f}(x) = \dfrac{1+\sin x}{1+2\sin x}$, $0 \leqslant x \leqslant \pi$, (3)

Solution

$$
\begin{aligned}
u = 1 + \sin x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \\
v = 1 + 2\sin x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\cos x
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{f}(x) = \frac{1+\sin x}{1+2\sin x} \Rightarrow \mathrm{f}'(x) &= \frac{(1+2\sin x)(\cos x) - (1+\sin x)(2\cos x)}{(1+2\sin x)^2} \\
\Rightarrow \mathrm{f}'(x) &= \frac{\cos x[(1+2\sin x) - (2+2\sin x)]}{(1+2\sin x)^2} \\
\Rightarrow \mathrm{f}'(x) &= \underline{\underline{\frac{-\cos x}{(1+2\sin x)^2}}}.
\end{aligned}
$$

(b) $\mathrm{g}(x) = \ln(1 + \mathrm{e}^{2x})$. (2)

Solution

$$
\begin{aligned}
\mathrm{g}(x) = \ln(1+\mathrm{e}^{2x}) \Rightarrow \mathrm{g}'(x) &= \frac{1}{1+\mathrm{e}^{2x}} \cdot \mathrm{e}^{2x} \cdot 2 \\
\Rightarrow \mathrm{g}'(x) &= \underline{\underline{\frac{2\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}}}.
\end{aligned}
$$

6. Given the curve (3)

$$
y = \frac{x}{x^2+4},
$$

calculate the gradient when $x = 2$.

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = x^2 + 4 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

$$y = \frac{x}{x^2 + 4} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2 + 4)(1) - (x)(2x)}{(x^2 + 4)^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 - x^2}{(x^2 + 4)^2}.$$

Finally,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 - 2^2}{(4^2 + 4)^2} = \underline{\underline{0}}.$$

7. Given that (2)

$$y = \sin(\mathrm{e}^{5x}),$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = \sin(\mathrm{e}^{5x}) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cos(\mathrm{e}^{5x}) \cdot 5\mathrm{e}^{5x}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^{5x}\cos(\mathrm{e}^{5x})}}.$$

8. Find the gradient of the tangent to the curve (4)

$$y = 2x\sqrt{x - 1}$$

at the point where $x = 10$.

Solution

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = (x-1)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}$$

Now,

$$y = 2x\sqrt{x-1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (2x)\left[\tfrac{1}{2}(x-1)^{-\frac{1}{2}}\right] + (2)\left[(x-1)^{\frac{1}{2}}\right]$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)^{-\frac{1}{2}}[x + 2(x-1)]$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)^{-\frac{1}{2}}(3x-2)$$

and

$$x = 10 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\underline{\tfrac{28}{3}}}.$$

9. Given that (3)

$$y = \mathrm{e}^{5x}\tan 2x,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$u = \mathrm{e}^{5x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 5\mathrm{e}^{5x}$$
$$v = \tan 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\sec^2 2x$$

$$y = \mathrm{e}^{5x}\tan 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (\mathrm{e}^{5x})(2\sec^2 2x) + (5\mathrm{e}^{5x})(\tan 2x)$$
$$\underline{\underline{\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{5x}(2\sec^2 2x + 5\tan 2x)}}.$$

10. A curve is defined by (5)

$$y = \frac{\sin x}{2 - \cos x} \text{ for } 0 \leqslant x \leqslant \pi.$$

Find the exact values of the coordinates of the stationary point of this curve.

Solution

$$u = \sin x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$
$$v = 2 - \cos x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$$

$$\begin{aligned} y = \frac{\sin x}{2 - \cos x} &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2 - \cos x)(\cos x) - (\sin x)(\sin x)}{(2 - \cos x)^2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2 - \cos x)^2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos x - 1}{(2 - \cos x)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{2\cos x - 1}{(2 - \cos x)^2} = 0 \\ &\Rightarrow 2\cos x - 1 = 0 \\ &\Rightarrow \cos x = \tfrac{1}{2} \\ &\Rightarrow x = \tfrac{1}{3}\pi \text{ (only)} \\ &\Rightarrow y = \tfrac{\sqrt{3}}{3}; \end{aligned}$$

hence, the exact values of the coordinate are $\underline{\underline{(\frac{1}{3}\pi, \frac{\sqrt{3}}{3})}}$.