Dr Oliver Mathematics
Applied Mathematics: Sequences

The total number of marks available is 25.
You must write down all the stages in your working.

1. Define
$$S_n = \sum_{r=1}^{n} r^2,\ n \geqslant 1.$$

(a) Write down formulae for S_n and S_{2n+1}. (2)

Solution

$$\begin{aligned} S_n &= \underline{\underline{\tfrac{1}{6}n(n+1)(2n+1)}} \\ S_{2n+1} &= \tfrac{1}{6}(2n+1)[(2n+1)+1][2(2n+1)+1] \\ &= \tfrac{1}{6}(2n+1)(2n+2)(4n+3) \\ &= \underline{\underline{\tfrac{1}{3}(n+1)(2n+1)(4n+3)}}. \end{aligned}$$

(b) Obtain a formula for (1)
$$2^2 + 4^2 + \ldots + (2n)^2.$$

Solution

$$\begin{aligned} 2^2 + 4^2 + \ldots + (2n)^2 &= (2 \cdot 1)^2 + (2 \cdot 2)^2 + \ldots + (2 \cdot n)^2 \\ &= (4 \cdot 1^2) + (4 \cdot 2^2) + \ldots + (4 \cdot n^2) \\ &= 4(1^2 + 2^2 + \ldots + n^2) \\ &= 4 \cdot \tfrac{1}{6}n(n+1)(2n+1) \\ &= \underline{\underline{\tfrac{2}{3}n(n+1)(2n+1)}}. \end{aligned}$$

2. (a) Use the standard formulas for (3)
$$\sum_{r=1}^{n} r \text{ and } \sum_{r=1}^{n} r^2$$
to show that
$$\sum_{r=1}^{n} (6r^2 - r) = \tfrac{1}{2}n(n+1)(4n+1).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(6r^2 - r) &= 6\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \\ &= n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) \\ &= \tfrac{1}{2}n(n+1)[2(2n+1) - 1] \\ &= \underline{\underline{\tfrac{1}{2}n(n+1)(4n+1)}},\end{aligned}$$

as required.

(b) Hence evaluate (2)

$$\sum_{r=5}^{10}(6r^2 - r).$$

Solution

$$\begin{aligned}\sum_{r=5}^{10}(6r^2 - r) &= \sum_{r=1}^{10}(6r^2 - r) - \sum_{r=1}^{4}(6r^2 - r) \\ &= \tfrac{1}{2}(10)(11)(41) - \tfrac{1}{2}(4)(5)(17) \\ &= 2\,255 - 170 \\ &= \underline{\underline{2\,085}}.\end{aligned}$$

3. (a) Find the value of N for which (3)

$$\sum_{r=1}^{N} r = 210.$$

Solution

$$\begin{aligned}\sum_{r=1}^{N} r = 210 &\Rightarrow \tfrac{1}{2}N(N+1) = 210 \\ &\Rightarrow N(N+1) = 420 \\ &\Rightarrow N^2 + N - 420 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +1 \\ \text{multiply to:} & -420\end{array}\right\} -20,\ +21$$

$$\begin{aligned}&\Rightarrow (N-20)(N+21)=0\\&\Rightarrow N-20=0 \text{ or } N+21=0\\&\Rightarrow N=20 \text{ or } N=-21;\end{aligned}$$

as $N > 0$, $\underline{\underline{N = 20}}$.

(b) Evaluate (2)

$$\sum_{r=1}^{N} r^2$$

for this value of N.

Solution

$$\begin{aligned}\sum_{r=1}^{20} r^2 &= \tfrac{1}{6}(20)(20+1)(2\cdot 20+1)\\&= \tfrac{1}{6}(20)(21)(41)\\&= \underline{\underline{2\,870}}.\end{aligned}$$

4. (a) State (2)

$$\sum_{r=1}^{n} r \text{ and } \sum_{r=1}^{n} r^3$$

in terms of n.

Solution

$$\sum_{r=1}^{n} r = \underline{\underline{\tfrac{1}{2}n(n+1)}} \text{ and } \sum_{r=1}^{n} r^3 = \underline{\underline{\tfrac{1}{4}n^2(n+1)^2}}.$$

(b) Hence show that (2)

$$\sum_{r=1}^{n}(r^3-3r) = \frac{n(n+1)(n-2)(n+3)}{4}.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^3-3r) &= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r \\ &= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{3}{2}n(n+1) \\ &= \tfrac{1}{4}n(n+1)\left[n(n+1)-6\right] \\ &= \tfrac{1}{4}n(n+1)\left[n^2+n-6\right]\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +1 \\ \text{multiply to:} & -6\end{array}\right\} -2,\ +3$$

$$\begin{aligned}&= \tfrac{1}{4}n(n+1)(n-2)(n+3) \\ &= \underline{\underline{\frac{n(n+1)(n-2)(n+3)}{4}}},\end{aligned}$$

as required,

(c) Use the above result to evaluate (2)

$$\sum_{r=5}^{15}(r^3-3r).$$

Solution

$$\begin{aligned}\sum_{r=5}^{15}(r^3-3r) &= \sum_{r=1}^{15}(r^3-3r) - \sum_{r=1}^{4}(r^3-3r) \\ &= \tfrac{1}{4}(15)(16)(13)(18) - \tfrac{1}{4}(4)(5)(2)(7) \\ &= 14\,040 - 70 \\ &= \underline{\underline{13\,970}}.\end{aligned}$$

5. Evaluate (2)

$$\sum_{r=1}^{80} 3r^2.$$

Solution

$$\begin{aligned}\sum_{r=1}^{80} 3r^2 &= 3\sum_{r=1}^{80} r^2 \\ &= 3 \times \tfrac{1}{6}(80)(81)(161) \\ &= 3 \times 173\,880 \\ &= \underline{\underline{521\,640}}.\end{aligned}$$

6. (a) Express (1)

$$\log_a 2 + \log_a 4 + \log_a 8$$

in the form

$$p \log_a 2,$$

where p is a constant.

Solution

$$\begin{aligned}\log_a 2 + \log_a 4 + \log_a 8 &= \log_a 2 + \log_a 2^2 + \log_a 2^3 \\ &= \log_a 2 + 2\log_a 2 + 3\log_a 2 \\ &= \underline{\underline{6\log_a 2}};\end{aligned}$$

hence, $\underline{\underline{p = 6}}$.

(b) Hence evaluate (3)

$$\sum_{r=1}^{100} \log_a 2^r,$$

giving your answer in the form

$$q \log_a 2,$$

where q is a constant.

Solution

$$\sum_{r=1}^{100} \log_a 2^r = \sum_{r=1}^{100} r \log_a 2$$
$$= \log_a 2 \sum_{r=1}^{100} r$$
$$= \log_a 2 \cdot \tfrac{1}{2}(100)(101)$$
$$= \underline{\underline{5\,050 \log_a 2}};$$

hence, $\underline{\underline{q = 5\,050}}$.