Dr Oliver Mathematics
Further Mathematics
Reduction Formulae
Past Examination Questions

This booklet consists of 24 questions across a variety of examination topics.
The total number of marks available is 281.

1.
$$I_n = \int_1^e x^2 (\ln x)^n \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (4)

$$I_n = \tfrac{1}{3}e^3 - \tfrac{1}{3}nI_{n-1}.$$

Solution

We use integration by parts:

$$u = (\ln x)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{n(\ln x)^{n-1}}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \tfrac{1}{3}x^3.$$

Hence

$$\begin{aligned} I_n &= \left[\tfrac{1}{3}x^3 (\ln x)^n\right]_{x=1}^{e} - \tfrac{n}{3}\int_1^e x^2 (\ln x)^{n-1} \, \mathrm{d}x \\ &= \left(\tfrac{1}{3}e^3 - 0\right) - \tfrac{n}{3}\int_1^e x^2 (\ln x)^{n-1} \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}e^3 - \tfrac{1}{3}nI_{n-1}}}, \end{aligned}$$

as required.

(b) Find the exact value of I_3. (4)

Solution

$$
\begin{aligned}
I_3 &= \tfrac{1}{3}e^3 - I_2 \\
&= \tfrac{1}{3}e^3 - \left(\tfrac{1}{3}e^3 - \tfrac{2}{3}I_1\right) \\
&= \tfrac{2}{3}I_1 \\
&= \tfrac{2}{3}\left(\tfrac{1}{3}e^3 - \tfrac{1}{3}I_0\right) \\
&= \tfrac{2}{9}e^3 - \tfrac{2}{9}I_0 \\
&= \tfrac{2}{9}e^3 - \tfrac{2}{9}\int_1^e x^2\,\mathrm{d}x \\
&= \tfrac{2}{9}e^3 - \tfrac{2}{9}\left[\tfrac{1}{3}x^3\right]_{x=1}^{e} \\
&= \tfrac{2}{9}e^3 - \tfrac{2}{9}\left(\tfrac{1}{3}e^3 - \tfrac{1}{3}\right) \\
&= \underline{\underline{\tfrac{2}{27} + \tfrac{4}{27}e^3}}.
\end{aligned}
$$

2.

$$I_n = \int_0^a (a-x)^n \cos x\,\mathrm{d}x,\ a > 0,\ n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (5)

$$I_n = na^{n-1} - n(n-1)I_{n-2}.$$

Solution

We use integration by parts:

$$u = (a-x)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -n(a-x)^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \Rightarrow v = \sin x.$$

Hence

$$
\begin{aligned}
I_n &= \left[(a-x)^n \sin x\right]_{x=0}^{a} + n\int_0^a (a-x)^{n-1}\sin x\,\mathrm{d}x \\
&= (0-0) + n\int_0^a (a-x)^{n-1}\sin x\,\mathrm{d}x \\
&= n\int_0^a (a-x)^{n-1}\sin x\,\mathrm{d}x \\
&= n\left\{\left[-(a-x)^{n-1}\cos x\right]_{x=0}^{a} - (n-1)\int_0^a (a-x)^{n-2}\cos x\,\mathrm{d}x\right\} \\
&= n\left\{0 - \left(-a^{n-1}\right) - (n-1)I_{n-2}\right\} \\
&= \underline{\underline{na^{n-1} - n(n-1)I_{n-2}}},
\end{aligned}
$$

as required

(b) Hence evaluate (3)

$$\int_0^{\frac{\pi}{2}} \left(\tfrac{\pi}{2} - x\right)^2 \, \mathrm{d}x.$$

Solution

We use part (a) with $a = \frac{\pi}{2}$:

$$\begin{aligned} I_2 &= 2 \times \tfrac{\pi}{2} - 2I_0 \\ &= \pi - 2\int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x \\ &= \pi - 2\left[\sin x\right]_{x=0}^{\frac{\pi}{2}} \\ &= \pi - 2(1 - 0) \\ &= \underline{\underline{\pi - 2}}. \end{aligned}$$

3.

$$I_n = \int x^n e^{2x} \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (3)

$$I_n = \tfrac{1}{2}(x^n e^{2x} - nI_{n-1}).$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x} \Rightarrow v = \tfrac{1}{2}e^{2x}.$$

Then

$$I_n = \tfrac{1}{2}x^n e^{2x} - \tfrac{1}{2}n \int x^{n-1} e^{2x} \, \mathrm{d}x = \underline{\underline{\tfrac{1}{2}(x^n e^{2x} - nI_{n-1})}},$$

as required.

(b) Find, in terms of e, the exact value of (5)

$$\int_0^1 x^2 e^{2x} \, \mathrm{d}x.$$

Solution

Let

$$J_n = \int_0^1 x^n e^{2x} \, \mathrm{d}x.$$

Then

$$\begin{aligned} J_n &= \left[\tfrac{1}{2} x^n e^{2x}\right]_{x=0}^1 - \tfrac{1}{2} n J_{n-1} \\ &= \left(\tfrac{1}{2} e^2 - 0\right) - \tfrac{1}{2} n J_{n-1} \\ &= \tfrac{1}{2} e^2 - \tfrac{1}{2} n J_{n-1}. \end{aligned}$$

So

$$\begin{aligned} J_2 &= \tfrac{1}{2} e^2 - J_1 \\ &= \tfrac{1}{2} e^2 - \left(\tfrac{1}{2} e^2 - \tfrac{1}{2} J_0\right) \\ &= \tfrac{1}{2} \int_0^1 e^{2x} \, \mathrm{d}x \\ &= \tfrac{1}{2} \left[\tfrac{1}{2} e^{2x}\right]_{x=0}^1 \\ &= \underline{\underline{\tfrac{1}{4}(e^2 - 1)}}. \end{aligned}$$

4.

$$I_n = \int_0^1 (1-x)^n \cosh x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (5)

$$I_n = n(n-1) I_{n-2} - n.$$

Solution

We use integration by parts:

$$u = (1-x)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -n(1-x)^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x \Rightarrow v = \sinh x.$$

Hence

$$\begin{aligned}
I_n &= [(1-x)^n \sinh x]_{x=0}^{1} + n\int_0^1 (1-x)^{n-1}\sinh x \,\mathrm{d}x \\
&= (0-0) + n\int_0^1 (1-x)^{n-1}\sinh x\,\mathrm{d}x \\
&= n\left\{\left[(1-x)^{n-1}\cosh x\right]_{x=0}^{1} + (n-1)\int_0^1 (1-x)^{n-2}\cosh x\,\mathrm{d}x\right\} \\
&= n\left\{(0-1) + (n-1)I_{n-2}\right\} \\
&= \underline{\underline{n(n-1)I_{n-2} - n}},
\end{aligned}$$

as required.

(b) Find an exact expression for I_4, giving your answer in terms of e. (3)

Solution

$$\begin{aligned}
I_4 &= 12I_2 - 4 \\
&= 12(2I_0 - 2) - 4 \\
&= 24I_0 - 28 \\
&= 24\int_0^1 \cosh x\,\mathrm{d}x - 28 \\
&= 24\left[\sinh x\right]_{x=0}^{1} - 28 \\
&= 24\sinh 1 - 28 \\
&= 24\left(\frac{e-e^{-1}}{2}\right) - 28 \\
&= \underline{\underline{12e - 12e^{-1} - 28}}.
\end{aligned}$$

5. Given that

$$I_n = \int_0^4 x^n\sqrt{4-x}\,\mathrm{d}x,\ n \geqslant 0,$$

(a) show that (6)

$$I_n = \frac{8n}{2n+3}I_{n-1},\ n \geqslant 1.$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = (4-x)^{\frac{1}{2}} \Rightarrow v = -\tfrac{2}{3}(4-x)^{\frac{3}{2}}.$$

So

$$\begin{aligned} I_n &= \left[-\tfrac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_{x=0}^{4} + \tfrac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,\mathrm{d}x \\ &= (0-0) + \tfrac{2}{3}n\int_0^4 x^{n-1}(4-x)(4-x)^{\frac{1}{2}}\,\mathrm{d}x \\ &= \tfrac{8}{3}nI_{n-1} - \tfrac{2}{3}nI_n. \end{aligned}$$

Then

$$3I_n = 8nI_{n-1} - 2nI_n \Rightarrow (2n+3)I_n = 8nI_{n-1} \Rightarrow I_n = \underline{\underline{\frac{8n}{2n+3}I_{n-1}}},$$

as required.

Given that

$$\int_0^4 \sqrt{4-x}\,\mathrm{d}x = \tfrac{16}{3},$$

(b) use the result in part (a) to find the exact value of (3)

$$\int_0^4 x^2\sqrt{4-x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned} I_2 &= \tfrac{16}{7}I_1 \\ &= \tfrac{16}{7}\times\tfrac{8}{5}I_0 \\ &= \tfrac{16}{7}\times\tfrac{8}{5}\times\tfrac{16}{3} \\ &= \underline{\underline{19\tfrac{53}{105}}}. \end{aligned}$$

6. Given that $y = \sinh^{n-1} x \cosh x$,

(a) show that (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (n-1)\sinh^{n-2}x + n\sinh^n x.$$

Solution

We use the product rule:

$$u = \sinh^{n-1}x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sinh^{n-2}x\cosh x \text{ and } v = \cosh x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \sinh x.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sinh^n x + (n-1)\sinh^{n-2}x\cosh^2 x \\ &= \sinh^n x + (n-1)\sinh^{n-2}x(1+\sinh^2 x) \\ &= \underline{\underline{(n-1)\sinh^{n-2}x + n\sinh^n x}},\end{aligned}$$

as required.

The integral I_n is defined by

$$I_n = \int_0^{\operatorname{arsinh} 1} \sinh^n x\,\mathrm{d}x,\ n \geqslant 0.$$

(b) Using the result in part (a), or otherwise, show that (2)

$$nI_n = \sqrt{2} - (n-1)I_{n-2},\ n \geqslant 2.$$

Solution

You probably don't like the look of the upper limit, arsinh 1, do you? Well, be of good cheer:

$$\sinh(\operatorname{arsinh} 1) = 1$$

and, since

$$\cosh^2(\operatorname{arsinh} 1) - \sinh^2(\operatorname{arsinh} 1) \equiv 1,$$

we have

$$\cosh(\operatorname{arsinh} 1) = \sqrt{1 + \sinh^2(\operatorname{arsinh} 1)} = \sqrt{1+1^2} = \sqrt{2}.$$

The trick is now to use the result in part (a) and integrate each side:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{n-1} x \cosh x\right) = (n-1)\sinh^{n-2} x + n\sinh^n x$$
$$\Rightarrow \left[\sinh^{n-1} x \cosh x\right]_{x=0}^{\operatorname{arsinh} 1} = (n-1)\int_0^{\operatorname{arsinh} 1} \sinh^{n-2} x \,\mathrm{d}x + n\int_0^{\operatorname{arsinh} 1} \sinh^n x \,\mathrm{d}x$$
$$\Rightarrow \sqrt{2} = (n-1)I_{n-2} + nI_n,$$

and hence

$$\underline{\underline{nI_n = \sqrt{2} - (n-1)I_{n-2}}},$$

as required.

(c) Hence find the exact value of I_4. (4)

Solution

$$\begin{aligned} I_4 &= \tfrac{1}{4}\left(\sqrt{2} - 3I_2\right) \\ &= \tfrac{1}{4}\sqrt{2} - \tfrac{3}{4} \times \tfrac{1}{2}\left(\sqrt{2} - I_0\right) \\ &= -\tfrac{1}{8}\sqrt{2} + \tfrac{3}{8}\int_0^{\operatorname{arsinh} 1} 1 \,\mathrm{d}x \\ &= -\tfrac{1}{8}\sqrt{2} + \tfrac{3}{8}(\operatorname{arsinh} 1 - 0) \\ &= \underline{\underline{-\tfrac{1}{8}\sqrt{2} + \tfrac{3}{8}\operatorname{arsinh} 1}}. \end{aligned}$$

7.

$$I_n = \int_1^5 x^n (2x-1)^{-\frac{1}{2}} \,\mathrm{d}x,\ n \geqslant 0.$$

(a) Prove that, for $n \geqslant 1$, (5)

$$(2n+1)I_n = nI_{n-1} + 3 \times 5^n - 1.$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = (2x-1)^{-\frac{1}{2}} \Rightarrow v = (2x-1)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} I_n &= \left[x^n(2x-1)^{\frac{1}{2}}\right]_{x=1}^{5} - n\int_1^5 x^{n-1}(2x-1)^{\frac{1}{2}}\,\mathrm{d}x \\ &= 3\times 5^n - 1 - n\int_1^5 x^{n-1}(2x-1)(2x-1)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= 3\times 5^n - 1 - 2n\int_1^5 x^{n}(2x-1)^{-\frac{1}{2}}\,\mathrm{d}x + n\int_1^5 x^{n-1}(2x-1)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= 3\times 5^n - 1 - 2nI_n + nI_{n-1}, \end{aligned}$$

and we can rearrange to get

$$\underline{\underline{(2n+1)I_n = nI_{n-1} + 3\times 5^n - 1}},$$

as required.

(b) Using the reduction formula given in part (a), find the exact value of I_2. (5)

Solution

$$I_0 = \int_1^5 (2x-1)^{-\frac{1}{2}}\,\mathrm{d}x = \left[(2x-1)^{\frac{1}{2}}\right]_{x=1}^{5} = 3 - 1 = 2.$$

Now

$$5I_2 = 2I_1 + 74 \Rightarrow I_2 = \tfrac{2}{5}I_1 + \tfrac{74}{5}$$

and

$$3I_1 = I_0 + 14 \Rightarrow I_1 = \tfrac{1}{3}I_0 + \tfrac{14}{3}.$$

Hence

$$I_2 = \tfrac{2}{5}\left(\tfrac{1}{3}\times 2 + \tfrac{14}{3}\right) + \tfrac{74}{5} = \underline{\underline{16\tfrac{14}{15}}}.$$

8.

$$I_n = \int (\ln x)^n\,\mathrm{d}x,\ n \geqslant 0.$$

(a) Show that (4)

$$I_n = x(\ln x)^{n-1} - nI_{n-1},\ n \geqslant 1.$$

Solution

We use integration by parts:

$$u = (\ln x)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{n(\ln x)^{n-1}}{x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x.$$

Hence

$$\begin{aligned} I_n &= x(\ln x)^n - \int \left(\frac{n(\ln x)^{n-1}}{x} \times x \right) \mathrm{d}x \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \, \mathrm{d}x \\ &= \underline{\underline{x(\ln x)^n - nI_{n-1}}}, \end{aligned}$$

as required.

(b) Hence calculate the exact value of (6)

$$\int_1^e (\ln x)^3 \, \mathrm{d}x.$$

Solution

I_n, for part (a), is an indefinite integral but here we are asked to find the exact value of a definite integral. My own preference, therefore, is to give this definite integral a different name and then to use the result of part (a) to set up a reduction formula. Let

$$J_n = \int_1^e (\ln x)^n \, \mathrm{d}x, \, n \geqslant 0.$$

Then, using part (a),

$$J_n = \left[x(\ln x)^{n-1} \right]_{x=1}^{e} - nJ_{n-1} = e - nJ_{n-1}.$$

So

$$
\begin{aligned}
J_3 &= e - 3J_2 \\
&= e - 3(e - 2J_1) \\
&= -2e + 6J_1 \\
&= -2e + 6(e - J_0) \\
&= 4e - 6J_0 \\
&= 4e - 6\int_1^e 1 \, \mathrm{d}x \\
&= 4e - 6(e - 1) \\
&= \underline{\underline{6 - 2e}}.
\end{aligned}
$$

9. Given that

$$I_n = \int_0^4 x^n \sqrt{16 - x^2} \, \mathrm{d}x \, n \geqslant 0,$$

(a) prove that, for $n \geqslant 2$, (6)

$$(n+2)I_n = 16(n-1)I_{n-2}.$$

Solution

We use integration by parts:

$$u = x^{n-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{16 - x^2} \Rightarrow v = -\tfrac{1}{3}(16 - x^2)^{\frac{3}{2}}.$$

Hence

$$
\begin{aligned}
I_n &= \left[-\tfrac{1}{3}x^{n-1}(16 - x^2)^{\frac{3}{2}}\right]_{x=0}^{4} + \tfrac{1}{3}(n-1)\int_0^4 x^{n-2}(16 - x^2)^{\frac{3}{2}} \, \mathrm{d}x \\
&= (0 - 0) + \tfrac{1}{3}(n-1)\int_0^4 x^{n-2}(16 - x^2)(16 - x^2)^{\frac{1}{2}} \, \mathrm{d}x \\
&= \tfrac{16}{3}(n-1)\int_0^4 x^{n-2}(16 - x^2)^{\frac{1}{2}} \, \mathrm{d}x - \tfrac{1}{3}(n-1)\int_0^4 x^n(16 - x^2)^{\frac{1}{2}} \, \mathrm{d}x \\
&= \tfrac{16}{3}(n-1)I_{n-2} - \tfrac{1}{3}(n-1)I_n.
\end{aligned}
$$

Hence

$$3I_n = 16(n-1)I_{n-2} - (n-1)I_n \Rightarrow \underline{\underline{(n+2)I_n = 16(n-1)I_{n-2}}},$$

as required.

(b) Hence, showing each step of your working, find the exact value of I_5. (5)

Solution

$$I_1 = \int_0^4 x\sqrt{16-x^2}\,\mathrm{d}x = \left[-\tfrac{1}{3}(16-x^2)^{\frac{3}{2}}\right]_{x=0}^4 = 0 - \left(-\tfrac{64}{3}\right) = \tfrac{64}{3}.$$

$$\begin{aligned} I_5 &= \tfrac{64}{7} I_3 \\ &= \tfrac{64}{7} \times \tfrac{32}{5} I_1 \\ &= \tfrac{2048}{35} \times \tfrac{64}{3} \\ &= \underline{\underline{\tfrac{131072}{105} \text{ or } 1248\tfrac{32}{105}}}. \end{aligned}$$

10.

$$I_n = \int_0^{\frac{\pi}{4}} x^n \sin 2x \,\mathrm{d}x,\ n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (5)

$$I_n = \tfrac{1}{4}n\left(\tfrac{\pi}{4}\right)^{n-1} - \tfrac{1}{4}n(n-1)I_{n-2}.$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \Rightarrow v = -\tfrac{1}{2}\cos 2x.$$

Hence

$$\begin{aligned} I_n &= \left[-\tfrac{1}{2}x^n \cos 2x\right]_0^{\frac{\pi}{4}} + \tfrac{1}{2}n \int_0^{\frac{\pi}{4}} x^{n-1}\cos 2x \,\mathrm{d}x \\ &= (0-0) + \tfrac{1}{2}n \int_0^{\frac{\pi}{4}} x^{n-1}\cos 2x\,\mathrm{d}x \\ &= \tfrac{1}{2}n\left\{\left[-\tfrac{1}{2}x^{n-1}\sin 2x\right]_0^{\frac{\pi}{4}} - \tfrac{1}{2}(n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\sin 2x\,\mathrm{d}x\right\} \\ &= \tfrac{1}{2}n\left(\tfrac{1}{2} \times \left(\tfrac{\pi}{4}\right)^{n-1} - 0\right) - \tfrac{1}{4}n(n-1)I_{n-2}, \\ &= \underline{\underline{\tfrac{1}{4}n\left(\tfrac{\pi}{4}\right)^{n-1} - \tfrac{1}{4}n(n-1)I_{n-2}}}, \end{aligned}$$

as required.

(b) Find the exact value of I_2. (4)

Solution

$$\begin{aligned} I_2 &= \tfrac{1}{4} \times 2 \times \tfrac{\pi}{4} - \tfrac{1}{4} \times 2 \times I_0 \\ &= \tfrac{1}{8}\pi - \tfrac{1}{2} \int_0^{\frac{\pi}{4}} \sin 2x \, \mathrm{d}x \\ &= \tfrac{1}{8}\pi - \tfrac{1}{2} \left[-\tfrac{1}{2} \cos 2x \right]_0^{\frac{\pi}{4}} \\ &= \tfrac{1}{8}\pi - \tfrac{1}{2} \left[0 - \left(-\tfrac{1}{2} \right) \right] \\ &= \underline{\underline{\tfrac{1}{8}\pi - \tfrac{1}{4}}}. \end{aligned}$$

(c) Show that $I_4 = \frac{1}{64}(\pi^3 - 24\pi + 48)$. (2)

Solution

$$\begin{aligned} I_4 &= \left(\tfrac{\pi}{4}\right)^3 - 3I_2 \\ &= \tfrac{1}{64}\pi^3 - 3\left(\tfrac{1}{8}\pi + \tfrac{1}{4}\right) \\ &= \underline{\underline{\tfrac{1}{64}\left(\pi^3 - 24\pi + 48\right)}}, \end{aligned}$$

as required.

11.

$$I_n = \int \sin^n x \, \mathrm{d}x, \; n \geqslant 0.$$

(a) Prove that, for $n \geqslant 2$, (4)

$$I_n = \tfrac{1}{n}\left(-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right).$$

Solution

We use integration by parts:

$$u = \sin^{n-1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sin^{n-2} x \cos x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x.$$

Hence

$$\begin{aligned} I_n &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, \mathrm{d}x \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, \mathrm{d}x \\ &= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n, \end{aligned}$$

and hence

$$nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} \Rightarrow \underline{\underline{I_n = \tfrac{1}{n}\left(-\sin^{n-1} x \cos x + (n-1) I_{n-2}\right)}},$$

as required.

Given that n is an odd number, $n \geqslant 3$,

(b) show that (4)

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \frac{(n-1)(n-3)\ldots 6 \times 4 \times 2}{n(n-2)(n-4)\ldots 7 \times 5 \times 3}.$$

Solution

Let

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x.$$

Then, using (a),

$$\begin{aligned} J_n &= \tfrac{1}{n}\left\{\left[-\sin^{n-1} x \cos x\right]_{x=0}^{\frac{\pi}{2}} + (n-1) J_{n-2}\right\} \\ &= \tfrac{1}{n}\left\{(0-0) + (n-1) J_{n-2}\right\} \\ &= \frac{(n-1)}{n} J_{n-2} \end{aligned}$$

and so

$$\begin{aligned} J_n &= \frac{(n-1)}{n} \times \frac{(n-3)}{(n-2)} J_{n-4} \\ &= \ldots = \frac{(n-1)(n-3)\ldots 6 \times 4 \times 2}{n(n-2)(n-4)\ldots 7 \times 5 \times 3} J_1. \end{aligned}$$

Finally,

$$J_1 = \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x = \left[-\cos x\right]_{x=0}^{\frac{\pi}{2}} = 0 - (-1) = 1,$$

and hence

$$\int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \underline{\underline{\frac{(n-1)(n-3)\ldots 6 \times 4 \times 2}{n(n-2)(n-4)\ldots 7 \times 5 \times 3}}},$$

as required.

(c) Hence find (3)

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^5 x(1-\sin^2 x)\,\mathrm{d}x \\ &= J_5 - J_7 \\ &= \frac{4\times 2}{5\times 3} - \frac{6\times 4\times 2}{7\times 5\times 3} \\ &= \tfrac{8}{15} - \tfrac{16}{35} \\ &= \underline{\underline{\tfrac{8}{105}}}.\end{aligned}$$

12. Given that

$$I_n = \int_0^{\pi} e^x \sin^n x \,\mathrm{d}x,\ n \geqslant 0,$$

(a) show that, for $n \geqslant 2$, (8)

$$I_n = \frac{n(n-1)}{n^2+1} I_{n-2}.$$

Solution

We use integration by parts:

$$u = \sin^n x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = n\cos x \sin^{n-1} x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^x \Rightarrow v = e^x.$$

Hence

$$\begin{aligned} I_n &= [e^x \sin^n x]_{x=0}^{\pi} - n \int_0^{\pi} e^x \cos x \sin^{n-1} x \, \mathrm{d}x \\ &= (0 - 0) - n \int_0^{\pi} e^x \cos x \sin^{n-1} x \, \mathrm{d}x \\ &= -n \int_0^{\pi} e^x \cos x \sin^{n-1} x \, \mathrm{d}x. \end{aligned}$$

Hmm. You may not like the look of this but that's too bad: we are going to have to use integration by parts here. But you need to make a sensible choice so that, first, your approach is actually do-able and, second, so that you are not merely reversing your steps.

$u = \sin^{n-1} x \cos x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\cos^2 x \sin^{n-2} x - \sin^n x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = e^x \Rightarrow v = e^x$

Then

$$\begin{aligned} I_n &= -n\left\{\left[e^x(\sin^{n-1} x \cos x)\right]_{x=0}^{\pi} - \int_0^{\pi} e^x \left((n-1)\cos^2 x \sin^{n-2} x - \sin^n x\right) \, \mathrm{d}x\right\} \\ &= -n\left\{(0-0) - (n-1)\int_0^{\pi} e^x \cos^2 x \sin^{n-2} x \, \mathrm{d}x + I_n\right\} \\ &= -n\left\{-(n-1)\int_0^{\pi} e^x (1 - \sin^2 x) \sin^{n-2} x \, \mathrm{d}x + I_n\right\} \\ &= n(n-1)I_{n-2} - n(n-1)I_n - nI_n \\ &= n(n-1)I_{n-2} - n^2 I_n. \end{aligned}$$

Hence

$$(n^2+1)I_n = n(n-1)I_{n-2} \Rightarrow I_n = \underline{\underline{\frac{n(n-1)}{n^2+1}I_{n-2}}},$$

as required.

(b) Find the exact value of I_4. (4)

Solution

$$
\begin{aligned}
I_4 &= \tfrac{12}{17} I_2 \\
&= \tfrac{12}{17} \times \tfrac{2}{5} I_0 \\
&= \tfrac{24}{85} \int_0^{\pi} e^x \, \mathrm{d}x \\
&= \tfrac{24}{85} \left[e^x \right]_{x=0}^{\pi} \\
&= \underline{\underline{\tfrac{24}{85}(e^{\pi} - 1)}}.
\end{aligned}
$$

13. Given that

$$I_n = \int_0^8 x^n (8 - x)^{\frac{1}{3}} \, \mathrm{d}x, \ n \geqslant 0,$$

(a) show that (6)

$$I_n = \frac{24n}{3n + 4} I_{n-1}, \ n \geqslant 1.$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = n x^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = (8 - x)^{\frac{1}{3}} \Rightarrow v = -\tfrac{3}{4}(8 - x)^{\frac{4}{3}}.$$

Hence

$$
\begin{aligned}
I_n &= \left[-\tfrac{3}{4} x^n (8 - x)^{\frac{4}{3}} \right]_{x=0}^{8} + \tfrac{3}{4} n \int_0^8 x^{n-1} (8 - x)^{\frac{4}{3}} \, \mathrm{d}x \\
&= (0 - 0) + \tfrac{3}{4} n \int_0^8 x^{n-1} (8 - x)(8 - x)^{\frac{1}{3}} \, \mathrm{d}x \\
&= 6n I_{n-1} - \tfrac{3}{4} n I_n.
\end{aligned}
$$

So

$$4 I_n = 24n I_{n-1} - 3n I_n \Rightarrow (3n + 4) I_n = 24n I_{n-1} \Rightarrow I_n = \underline{\underline{\frac{24n}{3n + 4} I_{n-1}}},$$

as required.

(b) Hence find the exact value of (6)

$$\int_0^8 x(x + 5)(8 - x)^{\frac{1}{3}} \, \mathrm{d}x.$$

Solution

$$\begin{aligned}\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\,\mathrm{d}x &= \int_0^8 x^2(8-x)^{\frac{1}{3}}\,\mathrm{d}x + 5\int_0^8 x(8-x)^{\frac{1}{3}}\,\mathrm{d}x \\ &= I_2 + 5I_1 \\ &= \tfrac{48}{10}I_1 + 5I_1 \\ &= \tfrac{49}{5}I_1 \\ &= \tfrac{49}{5}\times\tfrac{24}{7}I_0 \\ &= \tfrac{168}{5}\int_0^8 (8-x)^{\frac{1}{3}}\,\mathrm{d}x \\ &= \tfrac{168}{5}\left[-\tfrac{3}{4}(8-x)^{\frac{4}{3}}\right]_{x=0}^{8} \\ &= \tfrac{168}{5}\,(0-(-12)) \\ &= \underline{\underline{403\tfrac{1}{5}}}.\end{aligned}$$

14.

$$I_n = \int x^n \cosh x\,\mathrm{d}x,\ n \geqslant 0.$$

(a) Show that, for $n \geqslant 2$, (4)

$$I_n = x^n\sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}.$$

Solution

We use integration by parts:

$$u = x^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x \Rightarrow v = \sinh x.$$

So

$$\begin{aligned}I_n &= x^n\sinh x - n\int x^{n-1}\sinh x\,dx \\ &= x^n\sinh x - n\left\{x^{n-1}\cosh x - (n-1)\int x^{n-2}\cosh x\,\mathrm{d}x\right\} \\ &= x^n\sinh x - nx^{n-1}\cosh x + n(n-1)\int x^{n-2}\cosh x\,\mathrm{d}x \\ &= \underline{\underline{x^n\sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}}},\end{aligned}$$

as required.

(b) Hence show that (5)

$$I_4 = f(x)\sinh x + g(x)\cosh x + c,$$

where $f(x)$ and $g(x)$ are functions to be found, and c is an arbitrary constant.

Solution

$$\begin{aligned} I_4 &= x^4 \sinh x - 4x^3 \cosh x + 12 I_2 \\ &= x^4 \sinh x - 4x^3 \cosh x + 12\left(x^2 \sinh x - 2x \cosh x + 2I_0\right) \\ &= (x^4 + 12x^2)\sinh x + (-4x^3 - 24x)\cosh x + 24\int \cosh x \, \mathrm{d}x \\ &= \underline{\underline{(x^4 + 12x^2 + 24)\sinh x + (-4x^3 - 24x)\cosh x + c}}. \end{aligned}$$

(c) Find the exact value of (3)

$$\int x^4 \cosh x \, \mathrm{d}x,$$

giving your answer in terms of e.

Solution

Using (b),

$$\begin{aligned} \int x^4 \cosh x \, \mathrm{d}x &= \left[(x^4 + 12x^2 + 24)\sinh x + (-4x^3 - 24x)\cosh x\right]_{x=0}^{1} \\ &= (37 \sinh 1 - 28 \cosh 1) - (0 - 0) \\ &= 37\left(\frac{e - e^{-1}}{2}\right) - 28\left(\frac{e + e^{-1}}{2}\right) \\ &= \underline{\underline{\tfrac{9}{2}e - \tfrac{65}{2}e^{-1}}}. \end{aligned}$$

15. Given that

$$I_n = \int \frac{\sin nx}{\sin x} \, \mathrm{d}x, \; n \geqslant 1,$$

(a) prove that, for $n \geqslant 3$, (3)

$$I_n - I_{n-2} = \int 2\cos(n-1)x \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
I_n - I_{n-2} &= \int \frac{\sin nx}{\sin x}\,\mathrm{d}x - \int \frac{\sin(n-2)x}{\sin x}\,\mathrm{d}x \\
&= \int \frac{\sin nx - \sin(n-2)x}{\sin x}\,\mathrm{d}x \\
&= \int \frac{2\cos\frac{1}{2}(nx+(n-2)x)\sin\frac{1}{2}(nx-(n-2)x)}{\sin x}\,\mathrm{d}x \\
&= \int \frac{2\cos(n-1)x\sin x}{\sin x}\,\mathrm{d}x \\
&= \underline{\underline{\int 2\cos(n-1)x\,\mathrm{d}x}},
\end{aligned}$$

as required.

(b) Hence, showing each step of your working, find the exact value of (7)

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin 5x}{\sin x}\,\mathrm{d}x,$$

giving your answer in the form $\frac{1}{12}(a\pi + b\sqrt{3} + c)$, where a, b, and c are integers to be found.

Solution

Let

$$J_n = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin nx}{\sin x}\,\mathrm{d}x.$$

Now,

$$
\begin{aligned}
J_5 - J_3 &= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 2\cos 4x \,\mathrm{d}x \\
&= \left[\tfrac{1}{2}\sin 4x\right]_{x=\frac{\pi}{12}}^{\frac{\pi}{6}} \\
&= \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{3} \\
&= 0; \\
J_3 - J_1 &= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 2\cos 2x \,\mathrm{d}x \\
&= \left[\sin 2x\right]_{x=\frac{\pi}{12}}^{\frac{\pi}{6}} \\
&= \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}; \\
J_1 &= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 1 \,\mathrm{d}x \\
&= \left[x\right]_{x=\frac{\pi}{12}}^{\frac{\pi}{6}} \\
&= \tfrac{1}{12}\pi.
\end{aligned}
$$

Hence,

$$J_5 = \underline{\underline{\tfrac{1}{12}(\pi + 6\sqrt{3} - 6)}}.$$

16. (a) Find (2)

$$\int xe^{-\frac{1}{2}x^2} \,\mathrm{d}x.$$

Solution

$$\int xe^{-\frac{1}{2}x^2} \,\mathrm{d}x = \underline{\underline{-e^{-\frac{1}{2}x^2} + c}}$$

Given that

$$I_n = \int_0^1 x^n e^{-\frac{1}{2}x^2} \,\mathrm{d}x,$$

(b) prove that $I_n = (n-1)I_{n-2} - e^{-\frac{1}{2}}$, $n \geqslant 2$. (5)

Solution

We need integration by parts:

$$u = x^{n-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = xe^{-\frac{1}{2}x^2} \Rightarrow v = -e^{-\frac{1}{2}x^2}.$$

Then

$$\begin{aligned} I_n &= \left[-x^{n-1}e^{-\frac{1}{2}x^2}\right]_{x=0}^{1} - \int_0^1 -(n-1)x^{n-2}e^{-\frac{1}{2}x^2}\,\mathrm{d}x \\ &= \left(-e^{-\frac{1}{2}} - 0\right) + (n-1)\int_0^1 x^{n-2}e^{-\frac{1}{2}x^2}\,\mathrm{d}x \\ &= \underline{\underline{(n-1)I_{n-2} - e^{-\frac{1}{2}}}}, \end{aligned}$$

as required.

(c) find the value of I_5, leaving your answer in terms of e. (6)

Solution

$$\begin{aligned} I_5 &= 4I_3 - e^{-\frac{1}{2}} \\ &= 4\left(2I_1 - e^{-\frac{1}{2}}\right) - e^{-\frac{1}{2}} \\ &= 8I_1 - 5e^{-\frac{1}{2}} \\ &= 8\left[-e^{-\frac{1}{2}x^2}\right]_{x=0}^{1} - 5e^{-\frac{1}{2}} \text{ using (a)} \\ &= 8\left(-e^{-\frac{1}{2}} - (-1)\right) - 5e^{-\frac{1}{2}} \\ &= \underline{\underline{8 - 13e^{-\frac{1}{2}}}}. \end{aligned}$$

17.

$$I_n = \int \frac{\sin nx}{\sin x}\,\mathrm{d}x,\ n > 0,\ n \in \mathbb{Z}.$$

(a) By considering $I_{n+2} - I_n$, or otherwise, show that (6)

$$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n.$$

Solution

$$\begin{aligned}
I_{n+2} - I_n &= \int \frac{\sin(n+2)x}{\sin x}\,\mathrm{d}x - \int \frac{\sin nx}{\sin x}\,\mathrm{d}x \\
&= \int \frac{\sin(n+2)x - \sin nx}{\sin x}\,\mathrm{d}x \\
&= \int \frac{2\cos(n+1)x \sin x}{\sin x}\,\mathrm{d}x \\
&= 2\int \cos(n+1)x\,\mathrm{d}x \\
&= \frac{2\sin(n+1)x}{n+1},
\end{aligned}$$

and hence

$$I_{n+2} = \frac{2\sin(n+1)x}{n+1} + I_n,$$

as required.

Note: if you are worried about the absence of '$+c$' when the integration of $\cos(n+1)x$ was done you need not be since I_{n+2} and I_n are indefinite integrals and the constant will then be added and subtracted when the integrals are evaluated.

(b) Hence evaluate (7)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 6x}{\sin x}\,\mathrm{d}x,$$

giving your answer in the form $p\sqrt{2} + q\sqrt{3}$, where p and q are rational numbers to be found.

Solution

Let

$$J_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin nx}{\sin x}\,\mathrm{d}x.$$

Then, using part(a),

$$J_{n+2} = \left[\frac{2\sin(n+1)x}{n+1}\right]_{x=\frac{\pi}{4}}^{\frac{\pi}{3}} + J_n.$$

So

$$\begin{aligned}
J_6 &= \left[\tfrac{2}{5}\sin 5x\right]_{x=\frac{\pi}{4}}^{\frac{\pi}{3}} + J_4 \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} - \left(-\tfrac{\sqrt{2}}{2}\right)\right) + \left[\tfrac{2}{3}\sin 3x\right]_{x=\frac{\pi}{4}}^{\frac{\pi}{3}} + J_2 \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\right) + \tfrac{2}{3}\left(0 - \tfrac{\sqrt{2}}{2}\right) + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{\sin x}\,\mathrm{d}x \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\right) - \tfrac{\sqrt{2}}{3} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin x\cos x}{\sin x}\,\mathrm{d}x \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\right) - \tfrac{\sqrt{2}}{3} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\cos x\,\mathrm{d}x \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\right) - \tfrac{\sqrt{2}}{3} + \left[2\sin x\right]_{x=\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \tfrac{2}{5}\left(-\tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{2}}{2}\right) - \tfrac{\sqrt{2}}{3} + \left(\sqrt{3} - \sqrt{2}\right) \\
&= \underline{\underline{-\tfrac{17}{15}\sqrt{2} + \tfrac{4}{5}\sqrt{3}}}.
\end{aligned}$$

18.

$$I_n = \int \frac{x^n}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

(a) Show that $nI_n = x^{n-1}\sqrt{1+x^2} - (n-1)I_{n-2}$, $n \geqslant 2$. (7)

Solution

We use integration by parts:

$$u = x^{n-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x}{\sqrt{1+x^2}} \Rightarrow v = \sqrt{1+x^2}.$$

Then

$$\begin{aligned}
I_n &= x^{n-1}\sqrt{1+x^2} - \int (n-1)x^{n-2}\sqrt{1+x^2}\,\mathrm{d}x \\
&= x^{n-1}\sqrt{1+x^2} - (n-1)\int \frac{x^{n-2}(1+x^2)}{\sqrt{1+x^2}}\,\mathrm{d}x \\
&= x^{n-1}\sqrt{1+x^2} - (n-1)\int \frac{x^{n-2}+x^n}{\sqrt{1+x^2}}\,\mathrm{d}x \\
&= x^{n-1}\sqrt{1+x^2} - (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}$$

and hence

$$\underline{\underline{nI_n = x^{n-1}\sqrt{1+x^2} - (n-1)I_{n-2}}},$$

as required.

The curve C has equation

$$y^2 = \frac{x^2}{\sqrt{1+x^2}},\ y \geqslant 0.$$

The finite region, R, is bounded by C, the x-axis, and the lines with equation $x = 0$ and $x = 2$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid so formed, giving your answer in terms of π, surds, and natural logarithms. (7)

Solution

Let

$$J_n = \int_0^3 \frac{x^n}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

Then

$$\begin{aligned}\text{Volume} &= \pi\int_0^3 \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}x \\ &= \pi J_2 \\ &= \frac{\pi}{2}\left[x\sqrt{1+x^2}\right]_{x=0}^{3} - \frac{\pi}{2}J_0 \\ &= \frac{\pi}{2}\left(3\sqrt{10}-0\right) - \frac{\pi}{2}\int_0^3 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x \\ &= \frac{3\pi\sqrt{10}}{2} - \left[\operatorname{arsinh} x\right]_{x=0}^{3} \\ &= \frac{3\pi\sqrt{10}}{2} - (\operatorname{arsinh} 3 - \operatorname{arsinh} 0) \\ &= \underline{\underline{\frac{3\pi\sqrt{10}}{2} - \ln(3+\sqrt{10})}}.\end{aligned}$$

19. Given that $I_n = \displaystyle\int \sec^n x\,\mathrm{d}x$, (14)

(a) show that

$$(n-1)I_n = \tan x\sec^{n-2} x + (n-2)I_{n-2},\ n \geqslant 2.$$

Solution
We use integration by parts:

$$u = \sec^{n-2} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)\sec^{n-3} x \times \sec x \tan x = (n-2)\sec^{n-2} x \tan x$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x \Rightarrow v = \tan x.$$

Hence

$$\begin{aligned} I_n &= \tan x \sec^{n-2} x - \int (n-2)\sec^{n-2} x \tan^2 x \, \mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, \mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2) \int (\sec^n x - \sec^{n-2} x) \, \mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}, \end{aligned}$$

and hence

$$\underline{\underline{(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}}},$$

as required.

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} \sec^3 x \, \mathrm{d}x,$$

giving your answer in terms of natural logarithms and surds.

Solution
Let

$$J_n = \int_0^{\frac{\pi}{3}} \sec^3 x \, \mathrm{d}x.$$

Then, using (a),

$$(n-1)J_n = \left[\tan x \sec^{n-2} x\right]_0^{\frac{\pi}{3}} + (n-2)J_{n-2}.$$

So

$$
\begin{aligned}
2J_3 &= \left[\tan x \sec x\right]_0^{\frac{\pi}{3}} + J_1 \\
&= (2\sqrt{3} - 0) + \int_0^{\frac{\pi}{3}} \sec x \, \mathrm{d}x \\
&= 2\sqrt{3} + \left[\ln|\sec x + \tan x|\right]_{x=0}^{\frac{\pi}{3}} \\
&= 2\sqrt{3} + \ln(2 + \sqrt{3}) - \ln 1 \\
&= \underline{\underline{2\sqrt{3} + \ln(2 + \sqrt{3})}}
\end{aligned}
$$

20.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x.$$

(a) Show that (8)

$$I_n = \frac{n-1}{n} I_{n-2}, \; n \in \mathbb{Z}, \; n \geqslant 2.$$

Solution

We use integration by parts:

$$u = \sin^{n-1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sin^{n-2} x \cos x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x.$$

Then

$$
\begin{aligned}
I_n &= \left[-\cos x \sin^{n-1} x\right]_{x=0}^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -(n-1)\sin^{n-2} x \cos^2 x \, \mathrm{d}x \\
&= (0 - 0) + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, \mathrm{d}x \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, \mathrm{d}x - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \\
&= (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}
$$

and we can now rearrange:

$$nI_n = (n-1)I_{n-1} \Rightarrow \underline{\underline{I_n = \frac{n-1}{n} I_{n-2}}}.$$

(b) Hence evaluate (7)

$$\int_0^{\frac{\pi}{2}} \sin^6 x(1+\cos^2 x)\,\mathrm{d}x,$$

giving your answer as a multiple of π.

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^6 x(1+\cos^2 x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sin^6 x(2-\sin^2 x)\,\mathrm{d}x\\ &= 2I_6 - I_8\\ &= 2I_6 - \tfrac{7}{8}I_6 = \tfrac{9}{8}I_6\\ &= \tfrac{9}{8}\times\tfrac{5}{6}I_4 = \tfrac{15}{16}I_4\\ &= \tfrac{15}{16}\times\tfrac{3}{4}I_2 = \tfrac{45}{64}I_2\\ &= \tfrac{45}{64}\times\tfrac{1}{2}I_0 = \tfrac{45}{128}I_0\\ &= \tfrac{45}{128}\int_0^{\frac{\pi}{2}} 1\,\mathrm{d}x\\ &= \tfrac{45}{128}\times\tfrac{\pi}{2} = \underline{\underline{\tfrac{45}{256}\pi}}.\end{aligned}$$

21.

$$I_n = \int_0^1 (1-x^2)^n\,\mathrm{d}x,\ n \geqslant 0.$$

(a) Prove that $(2n+1)I_n = 2nI_{n-1}$, $n \geqslant 1$. (7)

Solution

We use integration by parts:

$$u = (1-x^2)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2nx(1-x^2)^{n-1} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x.$$

Hence

$$\begin{aligned}I_n &= \left[x(1-x^2)^{n-1}\right]_{x=0}^1 - \int_0^1 -2nx^2(1-x^2)^{n-1}\,\mathrm{d}x\\ &= (0-0) + 2n\int_0^1 \left[1-(1-x^2)\right](1-x^2)^{n-1}\,\mathrm{d}x\\ &= 2n\int_0^1 \left[(1-x^2)^{n-1} - (1-x^2)^n\right]\,\mathrm{d}x\\ &= 2nI_{n-1} - 2nI_n,\end{aligned}$$

and hence

$$\underline{\underline{(2n+1)I_n = 2nI_{n-1}}},$$

as required.

(b) Prove, by induction, that (8)

$$I_n \leqslant \left(\frac{2n}{2n+1}\right)^n,$$

for all $n \in \mathbb{Z}^+$.

Solution

$$\begin{aligned} I_1 &= \int_0^1 (1-x^2)\,\mathrm{d}x \\ &= \left[x - \tfrac{1}{3}x^3\right]_{x=0}^1 \\ &= (1-\tfrac{1}{3}) - (0-0) \\ &= \tfrac{2}{3} \\ &= \left(\tfrac{2\times 1}{2\times 1+1}\right)^1, \end{aligned}$$

and so the result is true for $n = 1$.

For the inductive step, we need to show that

$$\frac{2k}{2k+1} \leqslant \frac{2k+2}{2k+3} \qquad (*)$$

as this will get us the inequality that we need. This follows readily from the fact that

$$\begin{aligned} 4k^2 + 6k \leqslant 4k^2 + 6k + 2 &\Rightarrow 2k(2k+3) \leqslant (2k+1)(2k+2) \\ &\Rightarrow \frac{2k}{2k+1} \leqslant \frac{2k+2}{2k+3}. \end{aligned}$$

Suppose now that the result is true for some $n = k$, i.e.,

$$I_k \leqslant \left(\frac{2k}{2k+1}\right)^k.$$

Then

$$\begin{aligned}
I_{k+1} &= \frac{2(k+1)}{2(k+1)+1} I_k \text{ using part (a)} \\
&= \frac{2k+2}{2k+3} I_k \\
&\leqslant \frac{2k+2}{2k+3} \times \left(\frac{2k}{2k+1}\right)^k \text{ by the inductive hypothesis} \\
&\leqslant \frac{2k+2}{2k+3} \times \left(\frac{2k+2}{2k+3}\right)^k \text{ using } (*) \\
&= \left(\frac{2k+2}{2k+3}\right)^{k+1},
\end{aligned}$$

and so the result is true for $n = k + 1$.
Hence, by mathematical induction, the result is true for all $n \in \mathbb{Z}^+$.

22.

$$I_n = \int_0^{\text{arsinh}\, 1} \sinh^n x,\ \mathrm{d}x,\ n \in \mathbb{N}.$$

(a) Show that $nI_n = \sqrt{2} - (n-1)I_{n-2}$, $n \geqslant 2$. (9)

Solution
We use integration by parts:

$$u = \sinh^{n-1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sinh^{n-2} x \cosh x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sinh x \Rightarrow v = \cosh x.$$

Hence

$$\begin{aligned}
I_n &= \left[\cosh x \sinh^{n-1} x\right]_{x=0}^{\text{arsinh}\,1} - \int_{x=0}^{\text{arsinh}\,1} (n-1)\sinh^{n-2} x \cosh^2 x \,\mathrm{d}x \\
&= \left[\sqrt{1+\sinh^2 x}\sinh^{n-1} x\right]_{x=0}^{\text{arsinh}\,1} - (n-1)\int_{x=0}^{\text{arsinh}\,1} \sinh^{n-2} x(1+\sinh^2)x \,\mathrm{d}x \\
&= \sqrt{2} - (n-1)I_{n-2} - (n-1)I_n,
\end{aligned}$$

and hence

$$\underline{\underline{nI_n = \sqrt{2} - (n-1)I_{n-2}}},$$

as required.

(b) Evaluate (7)

$$\int_0^{\operatorname{arsinh} 1} \sinh^5 x,\, \mathrm{d}x,$$

leaving your answer in surd form.

Solution

$$\begin{aligned} I_5 &= \tfrac{1}{5}\left(\sqrt{2} - 4I_3\right) \\ &= \tfrac{1}{5}\sqrt{2} - \tfrac{4}{5} \times \tfrac{1}{3}\left(\sqrt{2} - 2I_1\right) \\ &= -\tfrac{1}{15}\sqrt{2} + \tfrac{8}{15} I_1 \\ &= -\tfrac{1}{15}\sqrt{2} + \tfrac{8}{15}\int_0^{\operatorname{arsinh} 1} \sinh x,\, \mathrm{d}x \\ &= -\tfrac{1}{15}\sqrt{2} + \tfrac{8}{15}\left[\cosh x\right]_{x=0}^{\operatorname{arsinh} 1} \\ &= -\tfrac{1}{15}\sqrt{2} + \tfrac{8}{15}\left[\sqrt{1 + \sinh^2 x}\right]_{x=0}^{\operatorname{arsinh} 1} \\ &= -\tfrac{1}{15}\sqrt{2} + \tfrac{8}{15}\left(\sqrt{2} - 1\right) \\ &= \underline{\underline{\tfrac{1}{15}\left(7\sqrt{2} - 8\right)}}. \end{aligned}$$

23.

$$I_n = \int_0^1 x^n \sqrt{1 - x^2}\, \mathrm{d}x,\ n \geqslant 0.$$

(a) Find the value of I_1. (3)

Solution

$$I_1 = \int_0^1 x\sqrt{1 - x^2}\, \mathrm{d}x = \left[-\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}}\right]_{x=0}^{1} = -\tfrac{1}{3}(0 - 1) = \underline{\underline{\tfrac{1}{3}}}$$

(b) Show that, for $n \geqslant 2$, (9)

$$(n + 2)I_n = (n - 1)I_{n-2}.$$

Solution

We use integration by parts:

$$u = x^{n-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (n - 1)x^{n-2} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{1 - x^2} \Rightarrow v = -\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}}.$$

Hence

$$\begin{aligned} I_n &= \left[-\tfrac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_{x=0}^{1} - \int_0^1 -\tfrac{1}{3}(n-1)x^{n-2}(1-x^2)^{\frac{3}{2}}\,\mathrm{d}x \\ &= (0-0) + \tfrac{1}{3}(n-1)\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}\,\mathrm{d}x \\ &= \tfrac{1}{3}(n-1)\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\,\mathrm{d}x \\ &= \tfrac{1}{3}(n-1)\int_0^1 x^{n-2}\sqrt{1-x^2}\,\mathrm{d}x - \tfrac{1}{3}(n-1)\int_0^1 x^n\sqrt{1-x^2}\,\mathrm{d}x \\ &= \tfrac{1}{3}(n-1)I_{n-2} - \tfrac{1}{3}(n-1)I_n. \end{aligned}$$

Hence

$$3I_n = (n-1)I_{n-2} - (n-1)I_n \Rightarrow \underline{\underline{(n+2)I_n = (n-1)I_{n-2}}},$$

as required.

(c) Hence find the exact value of (4)

$$\int_0^1 x^7\sqrt{1-x^2}\,\mathrm{d}x.$$

Solution

Using parts (a) and (b),

$$\begin{aligned} I_7 &= \tfrac{6}{9}I_5 \\ &= \tfrac{2}{3}\times\tfrac{4}{7}I_3 = \tfrac{8}{21}I_3 \\ &= \tfrac{8}{21}\times\tfrac{2}{5}I_1 \\ &= \underline{\underline{\tfrac{16}{315}}}. \end{aligned}$$

24. (16)

$$I_n = \int_0^\pi \sin^{2n} x\,\mathrm{d}x,\ n\in\mathbb{N}.$$

(a) Calculate I_0 in terms of π.

Solution

$$I_0 = \int_0^\pi 1\,\mathrm{d}x = \underline{\underline{\pi}}$$

(b) Show that

$$I_n = \frac{2n-1}{2n} I_{n-1},\ n \geqslant 1.$$

Solution

We use integration by parts:

$$u = \sin^{2n-1} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (2n-1)\sin^{2n-2} x \cos x$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

and so

$$\begin{aligned} I_n &= \left[-\sin^{2n-1} x \cos x\right]_0^{\pi} - \int_0^{\pi} (2n-1)\sin^{2n-2} x \cos x(-\cos x)\,\mathrm{d}x \\ &= (0-0) + (2n-1)\int_0^{\pi} \sin^{2n-2} x \cos^2 x\,\mathrm{d}x \\ &= (2n-1)\int_0^{\pi} \sin^{2n-2} x(1-\sin^2 x)\,\mathrm{d}x \\ &= (2n-1)\int_0^{\pi} \left(\sin^{2n-2} x - \sin^{2n} x\right)\,\mathrm{d}x \\ &= (2n-1)I_{n-1} - (2n-1)I_n \end{aligned}$$

Hence $2nI_n = (2n-1)I_{n-1}$ and so

$$\underline{\underline{I_n = \frac{2n-1}{2n} I_{n-1},}}$$

as required.

(c) Find I_3 in terms of π.

Solution

$I_3 = \frac{5}{6}I_2 = \frac{5}{6} \times \frac{3}{4}I_1 = \frac{5}{8}I_1 = \frac{5}{8} \times \frac{1}{2}I_0 = \underline{\underline{\frac{5}{16}\pi}}$

The picture shows the curve with polar equation $r = a\sin^3\theta$, $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

(d) Using your answer to part (c), or otherwise, find the exact area bounded by this curve.

Solution

$$\textbf{Area} = \tfrac{1}{2}\int_0^{\pi} (a\sin^3\theta)^2\,\mathrm{d}\theta = \tfrac{1}{2}a^2\int_0^{\pi} \sin^6\theta\,\mathrm{d}\theta = \tfrac{1}{2}a^2 I_3 = \underline{\underline{\tfrac{5}{32}\pi a^2}}$$