Dr Oliver Mathematics
Mathematics: Higher
2017 Paper 1: Non-Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. Functions f and g are defined on suitable domains by

$$\mathrm{f}(x) = 5x \text{ and } \mathrm{g}(x) = 2\cos x.$$

(a) Evaluate f(g(0)). (1)

Solution

$$\begin{aligned} \mathrm{f}(\mathrm{g}(0)) &= \mathrm{f}(2) \\ &= \underline{\underline{10}}. \end{aligned}$$

(b) Find an expression for g(f(x)). (2)

Solution

$$\begin{aligned} \mathrm{g}(\mathrm{f}(x)) &= \mathrm{g}(5x) \\ &= \underline{\underline{2\cos 5x}}. \end{aligned}$$

2. The point $P(-2, 1)$ lies on the circle (4)

$$x^2 + y^2 - 8x - 6y - 15 = 0.$$

Find the equation of the tangent to the circle at P.

Solution

$$\begin{aligned} x^2 + y^2 - 8x - 6y - 15 = 0 &\Rightarrow x^2 - 8x + y^2 - 6y = 15 \\ &\Rightarrow (x^2 - 8x + 16) + (y^2 - 6y + 9) = 15 + 16 + 9 \\ &\Rightarrow (x-4)^2 + (y-3)^2 = 40; \end{aligned}$$

hence, the centre of the circle is $C(4,3)$. Now,

$$\begin{aligned}\text{gradient of } CP &= \frac{3-1}{4-(-2)} \\ &= \frac{2}{6} \\ &= \tfrac{1}{3}\end{aligned}$$

and the gradient of the tangent is

$$-\frac{1}{\frac{1}{3}} = -3.$$

Finally, the equation of the tangent is

$$\begin{aligned}y - 1 = -3(x+2) &\Rightarrow y - 1 = -3x - 6 \\ &\Rightarrow \underline{\underline{y = -3x - 5}}.\end{aligned}$$

3. Given (2)

$$y = (4x-1)^{12},$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\begin{aligned}y = (4x-1)^{12} &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12(4x-1)^{11} \times 4 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 48(4x-1)^{11}}}.\end{aligned}$$

4. Find the value of k for which the equation (3)

$$x^2 + 4x + (k-5) = 0$$

has equal roots.

Solution

$$\begin{aligned}
`b^2 - 4ac = 0' &\Rightarrow 4^2 - 4 \times 1 \times (k-5) = 0 \\
&\Rightarrow 16 - 4(k-5) = 0 \\
&\Rightarrow 4[4 - (k-5)] = 0 \\
&\Rightarrow 4(9-k) = 0 \\
&\Rightarrow \underline{\underline{k = 9}}.
\end{aligned}$$

5. Vectors $\mathbf{u}$ and $\mathbf{v}$ are

$$\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ -8 \\ 6 \end{pmatrix}$$

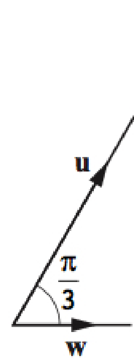
respectively.

(a) Evaluate $\mathbf{u}.\mathbf{v}$. (1)

Solution

$$\begin{aligned}
\mathbf{u}.\mathbf{v} &= \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} 3 \\ -8 \\ 6 \end{pmatrix} \\
&= 15 - 8 - 6 \\
&= \underline{\underline{1}}.
\end{aligned}$$

Vector $\mathbf{w}$ makes an angle of $\frac{1}{3}\pi$ with $\mathbf{u}$ and $|\mathbf{w}| = \sqrt{3}$.



(b) Calculate **u**.**w**. (3)

Solution

$$\begin{aligned}|\mathbf{u}| &= \sqrt{5^2 + 1^2 + (-1)^2} \\ &= \sqrt{27} \\ &= 3\sqrt{3}\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}.\mathbf{w} &= |\mathbf{u}||\mathbf{w}|\cos\theta \\ &= 3\sqrt{3}\cdot\sqrt{3}\cdot\cos\tfrac{1}{3}\pi \\ &= \underline{\underline{4\tfrac{1}{2}}}.\end{aligned}$$

6. A function, h, is defined by (3)

$$\mathrm{h}(x) = x^3 + 7, \text{ where } x \in \mathbb{R}.$$

Determine an expression for $h^{-1}(x)$.

Solution

$$\begin{aligned}y = x^3 + 7 \Rightarrow x^3 &= y - 7 \\ \Rightarrow x &= \sqrt[3]{y-7};\end{aligned}$$

hence,

$$\underline{\underline{h^{-1}(x) = \sqrt[3]{x-7}}}.$$

7. $A(-3, 5)$, $B(7, 9)$, and $C(2, 11)$ are the vertices of a triangle. (3)
Find the equation of the median through C.

Solution

The midpoint of AB is

$$D\left(\frac{-3+7}{2}, \frac{5+9}{2}\right) = (2, 7).$$

Well, $C(2, 11)$ and we are left with the equation of the median is $\underline{\underline{x = 2}}$.

8. Calculate the rate of change of (3)

$$\mathrm{d}(t) = \frac{1}{2t},\ t \neq 0,$$

when $t = 5$.

Solution

$$\mathrm{d}(t) = \frac{1}{2t} \Rightarrow \mathrm{d}(t) = \tfrac{1}{2}t^{-1}$$
$$\Rightarrow \mathrm{d}'(t) = -\tfrac{1}{2}t^{-2}.$$

Finally,

$$\mathrm{d}'(5) = -\tfrac{1}{2} \cdot 5^{-2} = \underline{\underline{-\tfrac{1}{50}}}.$$

9. A sequence is generated by the recurrence relation

$$u_{n+1} = mu_n + 6,$$

where m is a constant.

(a) Given $u_1 = 28$ and $u_2 = 13$, find the value of m. (2)

Solution

$$13 = 28m + 6 \Rightarrow 28m = 7$$
$$\Rightarrow \underline{\underline{m = \tfrac{1}{4}}}.$$

(b) (i) Explain why this sequence approaches a limit as $n \to \infty$. (1)

Solution
A limit exists the recurrence relation is <u>linear</u> and $\underline{\underline{|m| < 1}}$.

(ii) Calculate this limit. (2)

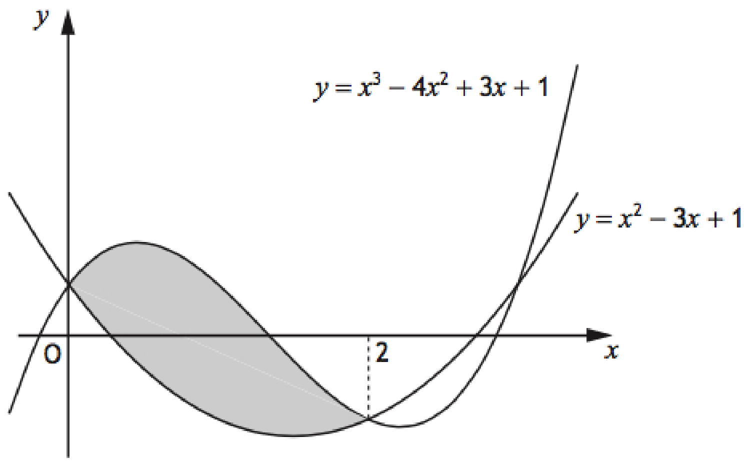
Solution
Let the limit be u. Then

$$u = \tfrac{1}{4}u + 6 \Rightarrow \tfrac{3}{4}u = 6$$
$$\Rightarrow \underline{\underline{u = 8}}.$$

10. Two curves with equations

$$y = x^3 - 4x^2 + 3x + 1 \text{ and } y = x^2 - 3x + 1$$

intersect as shown in the diagram.

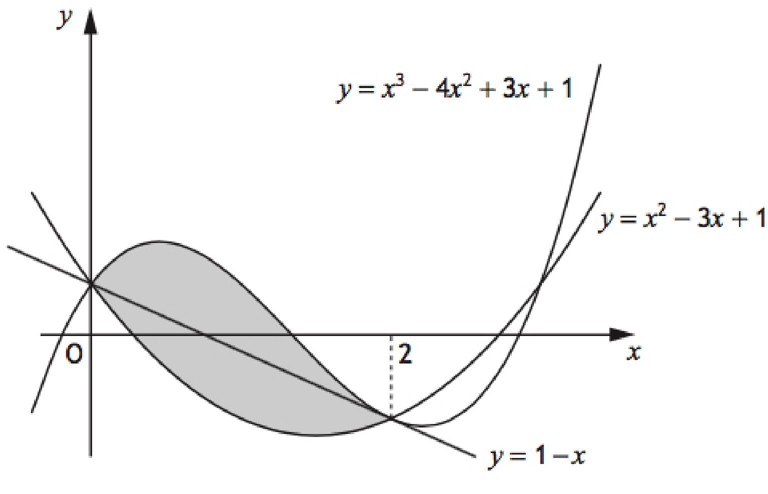


(a) Calculate the shaded area. (5)

Solution

$$\begin{aligned}\text{Shaded area} &= \int_0^2 [(x^3 - 4x^2 + 3x + 1) - (x^2 - 3x + 1)]\,\mathrm{d}x \\ &= \int_0^2 (x^3 - 5x^2 + 6x)\,\mathrm{d}x \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{5}{3}x^3 + 3x^2\right]_{x=0}^{2} \\ &= \left(4 - \tfrac{40}{3} + 12\right) - (0 - 0 + 0) \\ &= 16 - 13\tfrac{1}{3} \\ &= \underline{\underline{2\tfrac{2}{3}}}.\end{aligned}$$

The line passing through the points of intersection of the curves has equation $y = 1?x$.



(b) Determine the fraction of the shaded area which lies below the line $y = 1 - x$. (4)

Solution

$$\begin{aligned} \text{Fraction} &= \int_0^2 [(1-x) - (x^2 - 3x + 1)]\,\mathrm{d}x \\ &= \int_0^2 (2x - x^2)\,\mathrm{d}x \\ &= \left[x^2 - \tfrac{1}{3}x^3\right]_{x=0}^2 \\ &= \left(4 - \tfrac{8}{3}\right) - (0 - 0) \\ &= 1\tfrac{1}{3}; \end{aligned}$$

hence, the fraction is

$$\frac{1\frac{1}{3}}{2\frac{2}{3}} = \underline{\underline{\tfrac{1}{2}}}.$$

11. A and B are the points $(-7, 2)$ and $(5, a)$. (3)
AB is parallel to the line with equation

$$3y - 2x = 4.$$

Determine the value of a.

Solution

$$\begin{aligned} 3y - 2x &= 3(2) - 2(-7) \\ &= 6 + 14 \\ &= 20 \end{aligned}$$

and

$$\begin{aligned} 3a - 2(5) = 20 &\Rightarrow 3a = 30 \\ &\Rightarrow \underline{\underline{a = 10}}. \end{aligned}$$

12. Given that (3)

$$\log_a 36 - \log_a 4 = \tfrac{1}{2},$$

find the value of a.

Solution

$$\begin{aligned} \log_a 36 - \log_a 4 = \tfrac{1}{2} &\Rightarrow \log_a \left(\frac{36}{4}\right) = \tfrac{1}{2} \\ &\Rightarrow \log_a 9 = \tfrac{1}{2} \\ &\Rightarrow a^{\frac{1}{2}} = 9 \\ &\Rightarrow \underline{\underline{a = 81}}. \end{aligned}$$

13. Find (4)

$$\int \frac{1}{(5-4x)^{\frac{1}{2}}}\,\mathrm{d}x,\ x < \tfrac{5}{4}.$$

Solution

$$\begin{aligned}\int \frac{1}{(5-4x)^{\frac{1}{2}}}\,\mathrm{d}x &= \int (5-4x)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{(5-4x)^{\frac{1}{2}}}{\frac{1}{2}\cdot(-4)} + c \\ &= \underline{\underline{-\tfrac{1}{2}(5-4x)^{\frac{1}{2}} + c}}.\end{aligned}$$

14. (a) Express (4)

$$\sqrt{3}\sin x^\circ - \cos x^\circ$$

in the form

$$k\sin(x-a)^\circ,$$

where $k > 0$ and $0 < a < 360$.

Solution

$$\begin{aligned}\sqrt{3}\sin x^\circ - \cos x^\circ &\equiv k\sin(x-a)^\circ \\ &\equiv k(\sin x^\circ \cos a^\circ - \cos x^\circ \sin a^\circ) \\ &\equiv k\sin x^\circ \cos a^\circ - k\cos x^\circ \sin a^\circ\end{aligned}$$

and, hence,

$$k\cos a^\circ = \sqrt{3} \text{ and } k\sin a^\circ = 1.$$

Now,

$$\begin{aligned}k &= \sqrt{k^2} \\ &= \sqrt{k^2(\cos^2 a^\circ + \sin^2 a^\circ)} \\ &= \sqrt{(k\cos a^\circ)^2 + (k\sin a^\circ)^2} \\ &= \sqrt{(\sqrt{3})^2 + 1^2} \\ &= 2\end{aligned}$$

and

$$\begin{aligned}\tan a^\circ = \frac{\sin a^\circ}{\cos a^\circ} &\Rightarrow \tan a^\circ = \frac{1}{\sqrt{3}} \\ &\Rightarrow a = 30.\end{aligned}$$

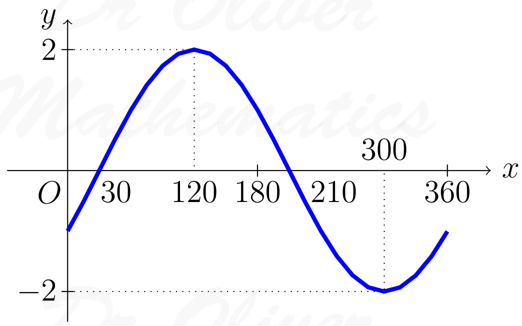
Hence,

$$\sqrt{3}\sin x^\circ - \cos x^\circ = \underline{\underline{2\sin(x-30)^\circ}}.$$

(b) Hence, or otherwise, sketch the graph with equation (3)

$$y = \sqrt{3}\sin x^\circ - \cos x^\circ,\ 0 \leqslant x \leqslant 360.$$

Solution



15. A quadratic function, f, is defined on $\mathbb{R}$, the set of real numbers.

Diagram 1 shows part of the graph with equation $y = \mathrm{f}(x)$.
The turning point is $(2, 3)$.

Diagram 2 shows part of the graph with equation $y = \mathrm{h}(x)$.
The turning point is $(7, 6)$.

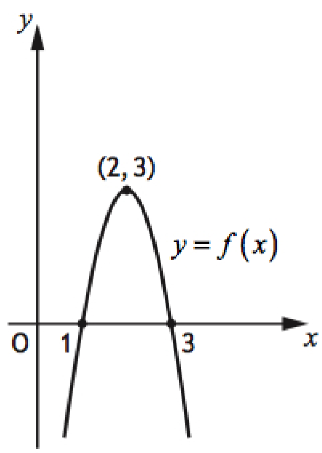


Diagram 1

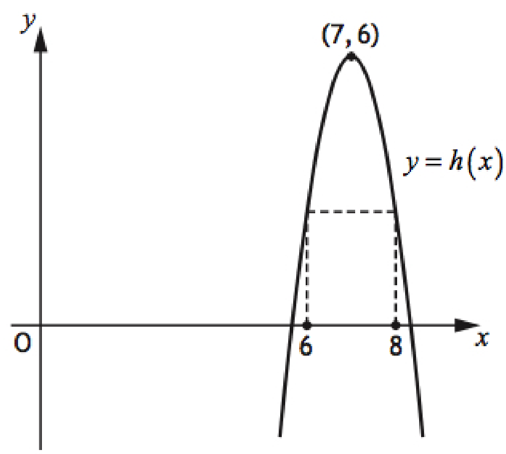


Diagram 2

Given that

$$\mathrm{h}(x) = \mathrm{f}(x + a) + b,$$

(a) write down the values of a and b. (2)

Solution
$\underline{\underline{a = -5}}$ and $\underline{\underline{b = 3}}$.

It is known that

$$\int_1^3 \mathrm{f}(x)\,\mathrm{d}x = 4.$$

(b) Determine the value of (1)

$$\int_6^8 \mathrm{h}(x)\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_6^8 \mathrm{h}(x)\,\mathrm{d}x &= \int_6^8 [\mathrm{f}(x-5)+3]\,\mathrm{d}x \\ &= \int_1^3 [\mathrm{f}(x)+3]\,\mathrm{d}x \\ &= \int_1^3 \mathrm{f}(x)\,\mathrm{d}x + \int_1^3 3\,\mathrm{d}x \\ &= 4 + [3x]_{x=1}^3 \\ &= 4 + (9-3) \\ &= \underline{\underline{10}}.\end{aligned}$$

(c) Given $\mathrm{f}'(1) = 6$, state the value of $\mathrm{h}'(8)$. (1)

Solution

$$\mathrm{h}(x) = \mathrm{f}(x-5) + 3 \Rightarrow \mathrm{h}'(x) = \mathrm{f}'(x-5)$$

and

$$\mathrm{h}'(8) = \mathrm{f}'(3) = \underline{\underline{-6}}$$

because

$$\mathrm{f}'(1) = 6 \Rightarrow \mathrm{f}'(3) = -6.$$