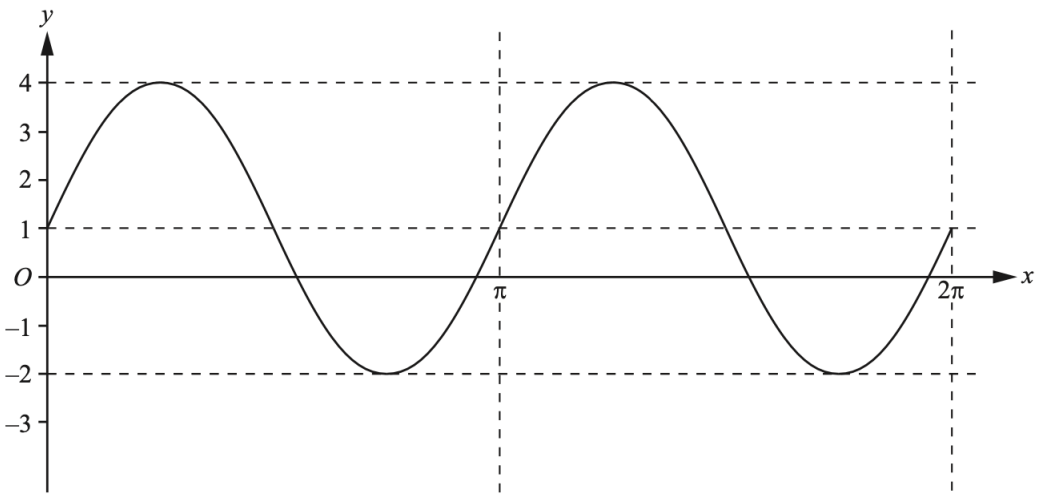
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 June Paper 1 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The diagram shows the graph of (3)

$$y = a\sin(bx + c), \text{ for } 0 \leqslant x \leqslant 2\pi,$$

where a, b, and c are positive integers.



State the values of a, b, and c.

Solution

Well,

$$a = \frac{4-(-2)}{2} = \underline{\underline{3}},$$

$$b = \underline{\underline{2}},$$

$$c = \underline{\underline{1}}.$$

2. Find the set of values of k for which the curve (4)

$$y = (k+1)x^2 - 3x + (k+1)$$

lies below the x-axis.

Solution

Quadratic formula: $a = k+1$, $b = -3$, and $c = k+1$:

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (-3)^2 - 4(k+1)(k+1) = 0 \\ &\Rightarrow 4(k+1)^2 = 9 \\ &\Rightarrow (k+1)^2 = \tfrac{9}{4} \\ &\Rightarrow k+1 = \pm\tfrac{3}{2} \\ &\Rightarrow k = -2\tfrac{1}{2} \text{ or } k = \tfrac{1}{2}. \end{aligned}$$

Thus, to be *below* the x-axis, we need $\underline{\underline{k < -2\tfrac{1}{2}}}$.

3. Show that (4)

$$\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \equiv 2\sec\theta.$$

Solution

Well,

$$\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \equiv \frac{(1+\sin\theta)^2}{\cos\theta(1+\sin\theta)} + \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$$

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$\times$	1	$+\sin\theta$
1	1	$+\sin\theta$
$+\sin\theta$	$+\sin\theta$	$+\sin^2\theta$

$$\begin{aligned}
&\equiv \frac{(1+2\sin\theta+\sin^2\theta)+\cos^2\theta}{\cos\theta(1+\sin\theta)}\\
&\equiv \frac{1+2\sin\theta+(\sin^2\theta+\cos^2\theta)}{\cos\theta(1+\sin\theta)}\\
&\equiv \frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}\\
&\equiv \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}\\
&\equiv \frac{2}{\cos\theta}\\
&\equiv \underline{\underline{2\sec\theta}},
\end{aligned}$$

as required.

4. The sets A and B are such that

$$A = \left\{x : \cos x = \tfrac{1}{2},\ 0^\circ \leqslant x \leqslant 620^\circ\right\},$$
$$B = \left\{x : \tan x = \sqrt{3},\ 0^\circ \leqslant x \leqslant 620^\circ\right\}.$$

(a) Find $\mathrm{n}(A)$. (1)

Solution

Well,

$$\cos x = \tfrac{1}{2} \Rightarrow x = 60,\ 300,\ 420,$$

and $\underline{\underline{\mathrm{n}(A) = 3}}$.

(b) Find $\mathrm{n}(B)$. (1)

Solution

Well,

$$\tan x = \sqrt{3} \Rightarrow x = 60,\ 240,\ 420,\ 600$$

and $\underline{\underline{\mathrm{n}(B) = 4}}$.

(c) Find the elements of $A \cup B$. (1)

Solution

$\underline{\underline{A \cup B = \{60,\ 240,\ 300,\ 420,\ 600\}}}$.

(d) Find the elements of $A \cap B$. (1)

Solution

$\underline{\underline{A \cap B = \{60,\ 420\}}}$.

5. (a) Find (3)

$$\int (9 + \sin 3x)\,\mathrm{d}x.$$

Solution

Well,

$$\int (9 + \sin 3x)\,\mathrm{d}x = \underline{\underline{9x - \tfrac{1}{3}\cos 3x + c}}.$$

(b) Hence show that (3)

$$\int_{-\frac{1}{9}\pi}^{\pi} (9 + \sin 3x)\,\mathrm{d}x = a\pi + b,$$

where a and b are constants to be found.

Solution

Now,

$$\begin{aligned}\int_{-\frac{1}{9}\pi}^{\pi} (9 + \sin 3x)\,\mathrm{d}x &= \left[9x - \tfrac{1}{3}\cos 3x\right]_{x=-\frac{1}{9}\pi}^{\pi} \\ &= \left(9\pi + \tfrac{1}{3}\right) - \left(\pi - \tfrac{1}{6}\right) \\ &= \underline{\underline{8\pi + \tfrac{1}{2}}};\end{aligned}$$

hence, $\underline{\underline{a = 8}}$ and $\underline{\underline{b = \tfrac{1}{2}}}$.

6. The function (6)

$$\mathrm{f}(x) = ax^3 + 4x^2 + bx - 2,$$

where a and b constants, is such that $(2x - 1)$ is a factor.

Given that the remainder when $\mathrm{f}(x)$ is divided by $(x - 2)$ is twice the remainder when $\mathrm{f}(x)$ is divided by $(x + 1)$, find the value of a and of b.

Solution

We use synthetic division:

$\frac{1}{2}$	a	4	b	-2
	$\downarrow$	$\frac{1}{2}a$	$\frac{1}{4}a + 2$	$\frac{1}{8}a + \frac{1}{2}b + 1$
	a	$\frac{1}{2}a + 4$	$\frac{1}{4}a + b + 2$	$\frac{1}{8}a + \frac{1}{2}b - 1$

so

$$\tfrac{1}{8}a + \tfrac{1}{2}b - 1 = 0 \Rightarrow \tfrac{1}{8}a + \tfrac{1}{2}b = 1 \quad (1).$$

Now,

2	a	4	b	-2
	$\downarrow$	$2a$	$4a + 8$	$8a + 2b + 16$
	a	$2a + 4$	$4a + b + 8$	$8a + 2b + 14$

and

-1	a	4	b	-2
	$\downarrow$	$-a$	$a - 4$	$-a - b + 4$
	a	$-a + 4$	$a + b - 4$	$-a - b + 2$

so

$$\begin{aligned} 8a + 2b + 14 = 2(-a - b + 2) &\Rightarrow 8a + 2b + 14 = -2a - 2b + 4 \\ &\Rightarrow 10a + 4b = -10 \quad (2). \end{aligned}$$

Do $8 \times (1)$:

$$a + 4b = 8 \quad (3)$$

and do (2) − (3):

$$9a = -18 \Rightarrow \underline{\underline{a = -2}}$$
$$\Rightarrow -2 + 4b = 8$$
$$\Rightarrow 4b = 10$$
$$\Rightarrow \underline{\underline{b = 2\tfrac{1}{2}}}.$$

7. (a) (i) Find how many different 4-digit numbers can be formed from the digits 1, 3, 5, 6, 8, and 9 if each digit may only be used only once. (1)

Solution

$$6 \times 5 \times 4 \times 3 = \underline{\underline{360}}.$$

(ii) Find how many of the 4-digit numbers are even. (1)

Solution

$$2 \times 5 \times 4 \times 3 = \underline{\underline{120}}.$$

(b) A team of 6 people is to be selected from 8 men and 4 women.

Find the number of different teams that can be selected if

(i) there are no restrictions, (1)

Solution

$$\binom{12}{6} = \underline{\underline{924}}.$$

(ii) the team contains all 4 women, (1)

Solution

$$\binom{8}{2} = \underline{\underline{28}}.$$

(iii) the team contains at least 4 men. (3)

Solution

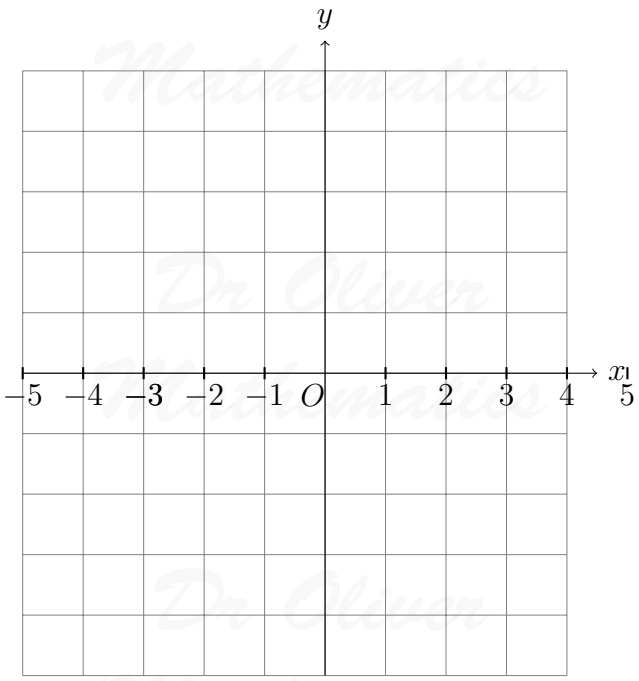
Well,

$$\begin{aligned}\text{at least 4 men} &= \text{4 men, 2 women} + \text{5 men, 1 woman} + \text{6 men}\\ &= \left[\binom{8}{4} \times \binom{4}{2}\right] + \left[\binom{8}{5} \times \binom{4}{1}\right] + \binom{8}{6}\\ &= 420 + 224 + 28\\ &= \underline{\underline{672}}.\end{aligned}$$

8. (a) On the grid, sketch the graph of (4)

$$y = |(x-2)(x+3)| \text{ for } -5 \leqslant x \leqslant 4,$$

and state the coordinates of the points where the curve meets the coordinate axis.



Solution

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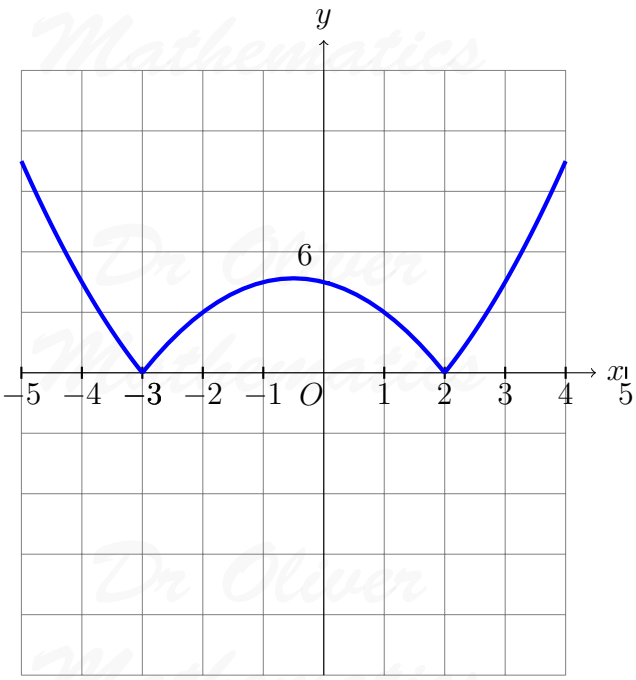

(b) Find the coordinates of the stationary points on the curve (2)

$$y = |(x-2)(x+3)|.$$

Solution

The x-coordinate is

$$\frac{-3+2}{2} = -0.5$$

and the y-coordinate is

$$|(-0.5-2)(-0.5+3)| = 6.25;$$

hence, the stationary point is $\underline{\underline{(-0.5, 6.25)}}$.

(c) Given that k is a positive constant, state the set of values of k for which (1)

$$|(x-2)(x+3)| = k$$

has only two solutions.

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Solution

$\underline{\underline{k > 6.25}}$.

9. (a) Differentiate (3)

$$4x^3 \ln(2x+1)$$

with respect to x.

Solution

Product rule:

$$u = 4x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 12x^2$$
$$v = \ln(2x+1) \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2}{2x+1}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4x^3)\left(\frac{2}{2x+1}\right) + (12x^2)(\ln(2x+1))$$
$$= \underline{\underline{\frac{8x^3}{2x+1} + 12x^2 \ln(2x+1).}}$$

(b) (i) Given that (4)

$$y = \frac{2x}{\sqrt{x+2}},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+4}{(\sqrt{x+2})^3}.$$

Solution

Quotient rule:

$$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$$
$$v = (x+2)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \tfrac{1}{2}(x+2)^{-\frac{1}{2}}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{((x+2)^{\frac{1}{2}})(2) - (\frac{1}{2}(x+2)^{-\frac{1}{2}})(2x)}{x+1} \\ &= \frac{2(x+2)^{\frac{1}{2}} - x(x+1)^{-\frac{1}{2}}}{x+2} \\ &= \frac{2(x+2)-x}{(x+1)^{\frac{3}{2}}} \\ &= \frac{2x+4-x}{(\sqrt{x+2})^3} \\ &= \underline{\underline{\frac{x+4}{(\sqrt{x+2})^3}}},\end{aligned}$$

as required.

(ii) Hence find (2)

$$\int \frac{5x+20}{(\sqrt{x+2})^3}\,\mathrm{d}x.$$

Solution
Well,

$$\begin{aligned}\int \frac{5x+20}{(\sqrt{x+2})^3}\,\mathrm{d}x &= \int \frac{5(x+4)}{(\sqrt{x+2})^3}\,\mathrm{d}x \\ &= 5\int \frac{x+4}{(\sqrt{x+2})^3}\,\mathrm{d}x \\ &= 5\left(\frac{2x}{\sqrt{x+2}}\right) + c \\ &= \underline{\underline{\frac{10x}{\sqrt{x+2}} + c}}.\end{aligned}$$

(iii) Hence evaluate (2)

$$\int_2^7 \frac{5x+20}{(\sqrt{x+2})^3}\,\mathrm{d}x.$$

Solution

Now,

$$\int_2^7 \frac{5x+20}{(\sqrt{x+2})^3}\,\mathrm{d}x = \left[\frac{10x}{\sqrt{x+2}}\right]_{x=2}^{7}$$
$$= 23\tfrac{1}{3} - 10$$
$$= \underline{\underline{13\tfrac{1}{3}}}.$$

10. **Solutions to this question by accurate drawing will not be accepted.**
The points $A(-3,2)$ and $B(1,4)$ are vertices of an isosceles triangle ABC, where angle $ABC = 90^\circ$.

(a) Find the length of the line AB. (1)

Solution

$$AB = \sqrt{[1-(-3)]^2 + (4-2)^2}$$
$$= \sqrt{4^2+2^2}$$
$$= \sqrt{20}$$
$$= \underline{\underline{2\sqrt{5}}}.$$

(b) Find the equation of the line BC. (3)

Solution

Well,

$$m_{AB} = \frac{4-2}{1-(-3)}$$
$$= \frac{2}{4}$$
$$= \tfrac{1}{2}$$

and

$$m_{BC} = -2.$$

Now, the equation of the line is

$$y - 4 = -2(x-1) \Rightarrow y - 4 = -2x + 2$$
$$\Rightarrow \underline{\underline{y = -2x + 6}}.$$

(c) Find the coordinates of each of the two possible positions of C. (6)

Solution

Well, the circle, centre B, radius $2\sqrt{5}$, passes through the point C:

$$\begin{aligned}(x-1)^2+(y-4)^2=20 &\Rightarrow (x-1)^2+(-2x+6-4)^2=20\\ &\Rightarrow (x-1)^2+(-2x+2)^2=20\end{aligned}$$

$\times$	x	-1
x	x^2	$-x$
-1	$-x$	$+1$

$\times$	$-2x$	$+2$
$-2x$	$4x^2$	$-4x$
$+2$	$-4x$	$+4$

$$\begin{aligned}&\Rightarrow (x^2-2x+1)+(4x^2-8x+4)=20\\ &\Rightarrow 5x^2-10x-15=0\\ &\Rightarrow 5(x^2-2x-3)=0\end{aligned}$$

add to: -2, multiply to: -3 $\Big\}$ $-3, +1$

$$\begin{aligned}&\Rightarrow 5(x-3)(x+1)=0\\ &\Rightarrow x-3=0 \text{ or } x+1=0\\ &\Rightarrow x=3 \text{ or } x=-1\\ &\Rightarrow y=0 \text{ or } y=8;\end{aligned}$$

hence, the two positions of C are $\underline{\underline{(3,0)}}$ and $\underline{\underline{(-1,8)}}$.

11. (a) It is given that the matrix

$$\mathbf{A}=\begin{pmatrix}2 & 3\\ 4 & 1\end{pmatrix}.$$

(i) Find $\mathbf{A} + 2\mathbf{I}$. (1)

Solution

$$\begin{aligned}\mathbf{A} + 2\mathbf{I} &= \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}}}.\end{aligned}$$

(ii) Find $\mathbf{A}^2$. (2)

Solution

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix}}}.\end{aligned}$$

(iii) Use your answers to parts (i) and (ii) to find the matrix $\mathbf{B}$ such that (2)

$$\mathbf{A}^2\mathbf{B} = \mathbf{I}.$$

Solution

Well, $\det \mathbf{A} = 208 - 108 = 100$ and

$$(\mathbf{A}^2)^{-1} = \tfrac{1}{100}\begin{pmatrix} 13 & -9 \\ -12 & 16 \end{pmatrix}.$$

Now,

$$\begin{aligned}\mathbf{A}^2\mathbf{B} = \mathbf{I} &= (\mathbf{A}^2)^{-1} \\ &= \underline{\underline{\tfrac{1}{100}\begin{pmatrix} 13 & -9 \\ -12 & 16 \end{pmatrix}}}.\end{aligned}$$

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(b) Given the matrix (4)

$$\mathbf{C} = \begin{pmatrix} x & -1 \\ x^2 - x + 1 & x - 1 \end{pmatrix},$$

show that $\det \mathbf{C} \neq 0$.

Solution
Well,

$$\begin{aligned} \det \mathbf{C} &= x(x-1) + (x^2 - x + 1) \\ &= (x^2 - x) + (x^2 - x + 1) \\ &= 2x^2 - 2x + 1 \\ &= 2(x^2 - x) + 1 \\ &= 2[(x^2 - x + \tfrac{1}{4}) - \tfrac{1}{4}] + 1 \\ &= 2(x - \tfrac{1}{2})^2 - \tfrac{1}{2} + 1 \\ &= 2(x - \tfrac{1}{2})^2 + \tfrac{1}{2} \\ &\underline{\underline{> 0}}, \end{aligned}$$

as required.

12. (a) A function f is such that

$$\mathrm{f}(x) = 3x^2 - 1 \text{ for } -10 \leqslant x \leqslant 8.$$

(i) Find the range of f. (3)

Solution
Well,

$$\begin{aligned} \mathrm{f}(-10) &= 299, \\ \mathrm{f}(0) &= -1, \\ \mathrm{f}(8) &= 191, \end{aligned}$$

so

$$\underline{\underline{-1 \leqslant \mathrm{f}(x) \leqslant 299}}.$$

(ii) Write down a suitable domain for f for which f^{-1} exists. (1)

Solution
$\underline{\underline{x \geqslant 0}}$.

(b) Function g and h are defined by

$$\mathrm{g}(x) = 4\mathrm{e}^x - 2 \text{ for } x \in \mathbb{R},$$
$$\mathrm{h}(x) = \ln 5x \text{ for } x > 0.$$

(i) Find $\mathrm{g}^{-1}(x)$. (2)

Solution
Well,

$$\begin{aligned} y = 4\mathrm{e}^x - 2 &\Rightarrow y + 2 = 4\mathrm{e}^x \\ &\Rightarrow \frac{y+2}{4} = \mathrm{e}^x \\ &\Rightarrow \ln\left(\frac{y+2}{4}\right) = x \end{aligned}$$

and so

$$\underline{\underline{\mathrm{g}^{-1}(x) = \ln\left(\frac{x+2}{4}\right)}}.$$

(ii) Solve (3)

$$\mathrm{g}\,\mathrm{h}(x) = 18.$$

Solution

$$\begin{aligned} \mathrm{g}\,\mathrm{h}(x) = 18 &\Rightarrow \mathrm{h}(x) = \mathrm{g}^{-1}(18) \\ &\Rightarrow \ln 5x = \ln\left(\frac{18+2}{4}\right) \\ &\Rightarrow \ln 5x = \ln 5 \\ &\Rightarrow 5x = 5 \\ &\Rightarrow \underline{\underline{x = 1}}. \end{aligned}$$

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