Dr Oliver Mathematics
Further Pure Mathematics
Maclaurin and Taylor Series
Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics. The total number of marks available is 203.

1. (a) Write down the expansion of $\sin 2x$ is ascending powers of x, up to and including the termin x^5. (1)

   (b) Show that, for some value of k, (4)

$$\lim_{x \to 0} \left[\frac{2x - \sin 2x}{x^2 \ln(1 + kx)} \right] = 16$$

   and state this value of k.

2. It is given that

$$y = \sqrt{4 + \sin x}.$$

   (a) Express $y \dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\cos x$. (2)

   (b) Find the value of $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ when $x = 0$. (5)

   (c) Hence, by using Maclaurin's theorem, find the first four terms in the expansion, is ascending powers of x of $\sqrt{4 + \sin x}$. (2)

3. (a)

$$y = \ln(\cos x + \sin x).$$

   (i) Show that (4)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{2}{1 + \sin 2x}.$$

   (ii) Find $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$. (1)

   (b) (i) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x + \sin x)$, are (3)

$$x - x^2 + \tfrac{2}{3}x^3.$$

   (ii) Write down the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\cos x - \sin x)$. (1)

(c) Hence find the first three non-zero terms in the expansion, in ascending powers of x, of **(4)**

$$\ln\left(\frac{\cos 2x}{\mathrm{e}^{3x-1}}\right).$$

4. (a) It is given that

$$y = \ln(\mathrm{e}^{3x}\cos x).$$

(i) Show that **(3)**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \tan x.$$

(ii) Find $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4}$. **(3)**

(b) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $y = \ln(\mathrm{e}^{3x}\cos x)$ are **(3)**

$$3x - \tfrac{1}{2}x^2 - \tfrac{1}{12}x^4.$$

(c) Write down the expansion of $\ln(1 + px)$, where p is a constant, in ascending powers of x up to and including the term in x^2. **(1)**

(d) (i) Find the value of p for which **(2)**

$$\lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{\mathrm{e}^{3x}\cos x}{1+px}\right)\right]$$

exists.

(ii) Hence find the value of **(2)**

$$\lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{\mathrm{e}^{3x}\cos x}{1+px}\right)\right]$$

when p takes the value found in part (d) (i).

5. Given that $y = \tan x$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, and $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$. **(3)**

(b) Find the Taylor series expansion of $\tan x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^3$. **(3)**

(c) Hence show that **(2)**

$$\tan\frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}.$$

6. (a) Prove by induction that (8)

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \cos x) = 2^{\frac{1}{2}n}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}n\pi),\ n \geqslant 1.$$

(b) Hence find the Maclaurin series expansion of $\mathrm{e}^x \cos x$, in ascending powers of x, up to and including the term in x^4. (3)

7. The variable y satisfies the differential equation

$$4(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

At $x = 0$, $y = 1$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$.

(a) Find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x = 0$. (1)

(b) Find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 0$. (4)

(c) Express y as a series, in ascending powers of x, up to and including the term in x^3. (2)

(d) Find the value that the series gives for y at $x = 0.1$, giving your answer to 5 decimal places. (1)

8.

$$(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + 4y^2.$$

(a) Show that (2)

$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x}. \quad (1)$$

(b) Differentiate equation (1) with respect to x to obtain an equation involving $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, x, y. (3)

Given that $y = \frac{1}{2}$ at $x = 0$,

(c) find a series solution for y, in ascending powers of x, up to and including the term in x^3. (6)

9. (a) Find the Taylor expansion of $\cos 2x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^5$. (5)

(b) Use your answer to (a) to obtain an estimate of $\cos 2$, giving your answer to 6 decimal places. (3)

10.

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

At $x = 0$, $y = 2$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$.

(a) Find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 0$. (3)

(b) Express y as a series in ascending powers of x, up to and including the term in x^3. (4)

11.

$$(x^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y^2 + (1-2x)\frac{\mathrm{d}y}{\mathrm{d}x}. \quad \text{(I)}$$

(a) By differentiating equation (I) with respect to x, show that (3)

$$(x^2+1)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = (1-4x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (4y-2)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ at $x = 0$,

(b) find the series solution for y, in ascending powers of x, up to and including the term in x^3. (4)

(c) Use your series to estimate the value of y at $x = -0.5$, giving your answer to two decimal places. (1)

12. Given that $y = x^3 \ln x$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (5)

(b) Find the Taylor series expansion of $x^3 \ln x$ in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$. (3)

13.

$$y = \sec^2 x.$$

(a) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.$$

(b) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^3$. (6)

14. The displacement x metres of a particle at time t seconds is given by the differential equation (5)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x + \cos x = 0.$$

When $t = 0$, $x = 0$, and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}$.

Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^3.

15.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right).$$

(a) Show that (3)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \mathrm{e}^x\left[2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + ky\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right],$$

where k is a constant to be found.

Given that, at $x = 0$, $y = 1$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$,

(b) find a series solution for y in ascending powers of x, up to and including the term in x^3. (4)

16.
$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 3x + y^2.$$

(a) Show that (2)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1 - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3.$$

Given that $y = 1$ at $x = 1$,

(b) find a series solution for y in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$. (8)

17. (5)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y - \sin x = 0$$

Given that $y = \frac{1}{2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{8}$ when $x = 0$, find a series expansion for y in terms of x, up to and including the term in x^3.

18. Given that
$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 5y = 0,$$

(a) Find $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ in terms of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and x. (4)

Given that $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$ at $x = 0$,

(b) find a series expansion for y in ascending powers of x, up to and including the term in x^3. (5)

19. (8)

$$y = \sqrt{8 + \mathrm{e}^x},\ x \in \mathbb{R}.$$

Find the series expansion for y in ascending powers of x, up to and including the term in x^2, giving each coefficient in its simplest form.

20.

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y = 0.$$

(a) Find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ in terms of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and y. (4)

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5$ at $x = 0$,

(b) find a series expansion for y in ascending powers of x, up to and including the term in x^3. (5)

21.

$$y = \tan^2 x,\ -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}.$$

(a) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.$$

(b) Hence show that (3)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 8\sec^2 x \tan x(A\sec^2 x + B),$$

where A and B are constants to be found.

(c) Find the Taylor series expansion of $\tan^2 x$, in ascending powers of $(x - \frac{\pi}{3})$, up to and including the term in $(x - \frac{\pi}{3})^3$. (4)

22. (a) Find the Taylor series expansion about $\frac{\pi}{4}$ of $\tan x$, in ascending powers of $(x - \frac{\pi}{4})$, up to and including the term in $(x - \frac{\pi}{4})^3$. (7)

(b) Deduce that an approximation for $\tan \frac{5\pi}{12}$ is (2)

$$1 + \tfrac{1}{3}\pi + \tfrac{1}{18}\pi^2 + \tfrac{1}{81}\pi^3.$$

23.

$$y = \ln\left(\frac{1}{1 - 2x}\right),\ |x| < \tfrac{1}{2}.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (4)

(b) Hence, or otherwise, find the series expansion of $\ln\left(\dfrac{1}{1-2x}\right)$ about $x = 0$, in ascending powers of x, up to and including the term in x^3. Give each coefficient in its simplest form. (3)

(c) Use your expansion to find an approximate value for $\ln\left(\frac{3}{2}\right)$, giving your answer to 3 decimal places. (3)