Dr Oliver Mathematics
Mathematics
Binomial Series
Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics.
The total number of marks available is 184.

For $n \in \mathbb{R}$,

$$\begin{aligned}(a+bx)^n &= \left[a(1+\tfrac{b}{a}x)\right]^n \\ &= a^n\left[1+\tfrac{b}{a}x\right]^n \\ &= a^n\left[1+n(\tfrac{b}{a}x)+\frac{n(n-1)}{2!}(\tfrac{b}{a}x)^2+\frac{n(n-1)(n-2)}{3!}(\tfrac{b}{a}x)^3+\dots\right]\end{aligned}$$

or

$$\boxed{(a+bx)^n = a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3+\dots}$$

provided $|x| < \dfrac{b}{a}$.

1. Use the binomial theorem to expand (5)

$$\sqrt{4-9x},\ |x|<\tfrac{4}{9},$$

in ascending powers of x, up to and including the term in x^3, simplifying each term.

Solution

$$\begin{aligned}\sqrt{4-9x} &= (4-9x)^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} + \tfrac{1}{2}(4^{-\frac{1}{2}})(-9x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4^{-\frac{3}{2}})(-9x)^2 \\ &\qquad + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(4^{-\frac{5}{2}})(-9x)^3+\dots \\ &= \underline{\underline{2-\tfrac{9}{4}x-\tfrac{81}{64}x^2-\tfrac{729}{512}x^3+\dots}}\end{aligned}$$

2.

$$f(x) \equiv \frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{A}{1-3x}+\frac{B}{2+x}+\frac{C}{(2+x)^2},\ |x|<\tfrac{1}{3}.$$

(a) Find the values of A and C and show that $B = 0$. (4)

Solution

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{A}{1-3x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}$$
$$\equiv \frac{A(2+x)^2 + B(1-3x)(2+x) + C(1-3x)}{(1-3x)(2+x)^2}$$

which means

$$3x^2 + 16 \equiv A(2+x)^2 + B(1-3x)(2+x) + C(1-3x).$$

$\underline{x=-2}$: $28 = 7C \Rightarrow C = 4$.
$\underline{x=\tfrac{1}{3}}$: $\frac{49}{3} = \frac{49}{9}A \Rightarrow A = 3$.
$\underline{x=0}$: $16 = 4A + 2B + C \Rightarrow B = 0$.
Hence

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \underline{\underline{\frac{3}{1-3x} + \frac{4}{(2+x)^2}}}.$$

(b) Hence, or otherwise, find the series expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3. Simplify each term. (7)

Solution

$$\frac{1}{1-3x} = (1-3x)^{-1}$$
$$= 1 + (-1)(-3x) + \frac{(-1)(-2)}{2!}(-3x)^2 + \frac{(-1)(-2)(-3)}{3!}(-3x)^3 + \ldots$$
$$= 1 + 3x + 9x^2 + 27x^3 + \ldots,$$

and

$$\frac{1}{(2+x)^2} = (2+x)^{-2}$$
$$= 2^{-2} + (-2)(2^{-3})x + \frac{(-2)(-3)}{2!}(2^{-4})x^2 + \frac{(-2)(-3)(-4)}{3!}(2^{-5})x^3 + \ldots$$
$$= \tfrac{1}{4} - \tfrac{1}{4}x + \tfrac{3}{16}x^2 - \tfrac{1}{8}x^3 \ldots,$$

and

$$\frac{3}{1-3x}+\frac{4}{(2+x)^2}$$
$$= 3(1+3x+9x^2+27x^3+\ldots)+4(\tfrac{1}{4}-\tfrac{1}{4}x+\tfrac{3}{16}x^2-\tfrac{1}{8}x^3\ldots)$$
$$= \underline{\underline{4+8x+\tfrac{111}{4}x^2+\tfrac{160}{2}x^3\ldots}}.$$

3.
$$\mathrm{f}(x)=\frac{3x-1}{(1-2x)^2},\ |x|<\tfrac{1}{2}.$$

Given that, for $x\neq\frac{1}{2}$,
$$\frac{3x-1}{(1-2x)^2}\equiv\frac{A}{1-2x}+\frac{B}{(1-2x)^2},$$
where A and B are constants,

(a) find the values of A and B. (3)

Solution

$$\frac{3x-1}{(1-2x)^2}\equiv\frac{A}{1-2x}+\frac{B}{(1-2x)^2}$$
$$\equiv\frac{A(1-2x)+B}{(1-2x)^2},$$

and so
$$3x-1\equiv A(1-2x)+B.$$

$\underline{x=\frac{1}{2}}$: $\frac{1}{2}=B$.
$\underline{x=0}$: $-1=A+B\Rightarrow A=-\frac{3}{2}$.
Hence
$$\frac{3x-1}{(1-2x)^2}\equiv\underline{\underline{\frac{-\frac{3}{2}}{1-2x}+\frac{\frac{1}{2}}{(1-2x)^2}}}.$$

(b) Hence, or otherwise, find the series expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3, simplifying each term. (6)

Solution

$$
\begin{aligned}
\frac{1}{1-2x} &= (1-2x)^{-1} \\
&= 1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \frac{(-1)(-2)(-3)}{3!}(-2x)^3 + \dots \\
&= 1 + 2x + 4x^2 + 8x^3 + \dots,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(1-2x)^2} &= (1-2x)^{-2} \\
&= 1 + (-2)(-2x) + \frac{(-2)(-3)}{2!}(-2x)^2 + \frac{(-2)(-3)(-4)}{3!}(-2x)^3 + \dots \\
&= 1 + 4x + 12x^2 + 32x^3 + \dots,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{-\frac{3}{2}}{1-2x} + \frac{\frac{1}{2}}{(1-2x)^2} \\
= \quad & -\tfrac{3}{2}(1 + 2x + 4x^2 + 8x^3 + \dots) + \tfrac{1}{2}(1 + 4x + 12x^2 + 32x^3 + \dots) \\
= \quad & \underline{\underline{-1 - x + 4x^3 + \dots}}
\end{aligned}
$$

4. (5)

$$\mathrm{f}(x) = (2-5x)^{-2},\ |x| < \tfrac{2}{5}.$$

Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3, giving each coefficient as a simplified fraction.

Solution

$$
\begin{aligned}
(2-5x)^{-2} &= 2^{-2} + (-2)(2^{-3})(-5x) + \frac{(-2)(-3)}{2!}(2^{-4})(-5x)^2 \\
&\qquad + \frac{(-2)(-3)(-4)}{3!}(2^{-5})(-5x)^3 + \dots \\
&= \underline{\underline{\tfrac{1}{4} + \tfrac{5}{4}x + \tfrac{75}{16}x^2 + \tfrac{125}{8}x^3 \dots}}
\end{aligned}
$$

5. (5)

$$\mathrm{f}(x) = (3+2x)^{-3},\ |x| < \tfrac{3}{2}.$$

Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3.

Give each coefficient as a simplified fraction.

Solution

$$(3+2x)^{-3} = 3^{-3} + (-3)(3^{-4})(2x) + \frac{(-3)(-4)}{2!}(3^{-5})(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(3^{-6})(2x)^3 + \dots$$
$$= \underline{\underline{\tfrac{1}{27} - \tfrac{2}{27}x + \tfrac{8}{81}x^2 - \tfrac{80}{729}x^3 \dots}}$$

6. (a) Use the binomial series to expand (5)

$$(8-3x)^{\frac{1}{3}},\ |x| < \tfrac{8}{3},$$

in ascending powers of x, up to and including the term in x^3, giving each terms as a simplified fraction.

Solution

$$(8-3x)^{\frac{1}{3}}$$
$$= \left[8(1-\tfrac{3}{8}x)\right]^{\frac{1}{3}}$$
$$= 2\left[1 + \tfrac{1}{3}\left(-\tfrac{3}{8}x\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\tfrac{3}{8}x\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(-\tfrac{3}{8}x\right)^3 + \dots\right]$$
$$= \underline{\underline{2 - \tfrac{1}{4}x - \tfrac{1}{32}x^2 - \tfrac{5}{768}x^3 + \dots}}$$

(b) Use your expansion, with a suitable value of x to obtain an approximation to $\sqrt[3]{7.7}$. Give your answer to 7 decimal places. (2)

Solution

$$\sqrt[3]{7.7} = \sqrt[3]{8 - 3\times 0.1}$$
$$\approx 2 - \tfrac{1}{4}(0.1) - \tfrac{1}{32}(0.1)^2 - \tfrac{5}{768}(0.1)^3$$
$$= 1.974\,680\,989\,6 \text{ (FCD)}$$
$$= \underline{\underline{1.974\,681\,0 \text{ (7 dp)}}}$$

7. (a) Expand $\dfrac{1}{\sqrt{4-3x}}$, where $|x| < \frac{4}{3}$, in ascending powers of x, up to and including the term in x^2. Simplify each term. (5)

Solution

$$\begin{aligned}\frac{1}{\sqrt{4-3x}} &= (4-3x)^{-\frac{1}{2}} \\ &= 4^{-\frac{1}{2}} + (-\tfrac{1}{2})(4^{-\frac{3}{2}})(-3x) + \frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)}{2!}(4^{-\frac{5}{2}})(-3x)^2 \\ &= \underline{\underline{\tfrac{1}{2} + \tfrac{3}{16}x + \tfrac{27}{256}x^2 + \ldots}}\end{aligned}$$

(b) Hence, or otherwise, find the first 3 term in the expansion of $\dfrac{x+8}{\sqrt{4-3x}}$ as a series in ascending powers of x. (4)

Solution

	$\frac{1}{2}$	$+\frac{3}{16}x$	$+\frac{27}{256}x^2$
x	$\frac{1}{2}x$	$+\frac{3}{16}x^2$	$\ldots$
$+8$	$+4$	$+\frac{3}{2}x$	$+\frac{27}{32}x^2$

So

$$\frac{x+8}{\sqrt{4-3x}} = \underline{\underline{4 + 2x + \tfrac{33}{32}x^2 + \ldots}}.$$

8.

$$\mathrm{f}(x) = \frac{27x^2 + 32x + 16}{(3x+2)^2(1-x)}, \; |x| < \tfrac{2}{3}.$$

Given that f(x can be expressed in the form

$$\mathrm{f}(x) = \frac{A}{3x+2} + \frac{B}{(3x+2)^2} + \frac{C}{1-x},$$

(a) find the values of B and C and show that $A = 0$. (4)

Solution

$$\frac{27x^2+32x+16}{(3x+2)^2(1-x)} \equiv \frac{A}{3x+2} + \frac{B}{(3x+2)^2} + \frac{C}{1-x}$$
$$\equiv \frac{A(3x+2)(1-x)+B(1-x)+C(3x+2)^2}{(3x+2)^2(1-x)}$$

and so

$$27x^2+32x+16 \equiv A(3x+2)(1-x)+B(1-x)+C(3x+2)^2.$$

$\underline{x=1}$: $75 = 25C \Rightarrow C = 3$.
$\underline{x=-\frac{2}{3}}$: $\frac{20}{3} = \frac{5}{3}B \Rightarrow B = 4$.
$\underline{x=0}$: $16 = 2A + B + 4C \Rightarrow A = 0$.
So

$$\frac{27x^2+32x+16}{(3x+2)^2(1-x)} \equiv \underline{\underline{\frac{4}{(3x+2)^2} + \frac{3}{1-x}}}.$$

(b) Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^2. Simplify each term. (6)

Solution

$$(2+3x)^{-2} = 2^{-2} + (-2)(2^{-3})(3x) + \frac{(-2)(-3)}{2!}(2^{-4})(3x)^2$$
$$= \tfrac{1}{4} - \tfrac{3}{4}x + \tfrac{27}{16}x^2 + \ldots,$$

$$\frac{1}{1-x} = (1-x)^{-1}$$
$$= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots$$
$$= 1 + x + x^2 + \ldots,$$

and

$$\frac{4}{(3x+2)^2} + \frac{3}{1-x}$$
$$\Rightarrow \quad 4(\tfrac{1}{4} - \tfrac{3}{4}x + \tfrac{27}{16}x^2 + \ldots) + 3(1 + x + x^2 + \ldots)$$
$$\Rightarrow \quad \underline{\underline{4 + \tfrac{39}{4}x^2 + \ldots}}$$

(c) Find the percentage error made in using the series expansion in part (b) to estimate (4) the value of $f(0.2)$. Give your answer to 2 significant figures.

Solution

Now,

$$f(0.2) = \tfrac{2935}{676}$$

and from the series

$$f(0.2) \approx 4.39$$

and this gives us

$$\begin{aligned}\%\text{ error} &= \frac{\left|\frac{2935}{676} - 4.39\right|}{\frac{2935}{676}} \times 100\% \\ &= 1.112\,095\,4 \text{ (FCD)} \\ &= \underline{\underline{1.1 \text{ (2 sf)}}}.\end{aligned}$$

9. (6)

$$f(x) = \frac{1}{\sqrt{4+x}}.$$

Find the binomial expansion of $f(x)$, in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

Solution

$$\begin{aligned}\frac{1}{\sqrt{4+x}} &= (4+x)^{-\frac{1}{2}} \\ &= 4^{-\frac{1}{2}} + (-\tfrac{1}{2})(4^{-\frac{3}{2}})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(4^{-\frac{5}{2}})x^2 \\ &\qquad + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(4^{-\frac{7}{2}})x^3 + \ldots \\ &= \underline{\underline{\tfrac{1}{2} - \tfrac{1}{16}x + \tfrac{3}{256}x^2 - \tfrac{5}{2048}x^3 + \ldots}}\end{aligned}$$

10. (a) Find the binomial expansion of (4)

$$(1-8x)^{\frac{1}{2}},\ |x| < \tfrac{1}{8},$$

in ascending powers of x, up to and including the term in x^3, simplifying each term.

Solution

$$\begin{aligned}\sqrt{1-8x} &= [1+(-8x)]^{\frac{1}{2}} \\ &= 1+\tfrac{1}{2}(-8x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-8x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-8x)^3+\ldots \\ &= \underline{\underline{1-4x-8x^2-32x^3+\ldots}}\end{aligned}$$

(b) Find the exact value of $(1-8x)^{\frac{1}{2}}$ when $x=0.01$. (2)

Solution

$$(1-8\times 0.01)^{\frac{1}{2}}=(0.92)^{\frac{1}{2}}=\underline{\underline{\tfrac{1}{5}\sqrt{23}}}.$$

(c) Hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places. (3)

Solution

Using parts (a) and (b),

$$\begin{aligned}\sqrt{23} &\approx 5\left[1-4(0.01)-8(0.01)^2-32(0.01)^3\right] \\ &= \underline{\underline{4.795\,84\text{ (5 dp)}}}.\end{aligned}$$

11.

$$\frac{2x^2+5x-10}{(x-1)(x+2)}\equiv A+\frac{B}{x-1}+\frac{C}{x+2}.$$

(a) Find the values of the constants A, B, and C. (4)

Solution

$$\begin{aligned}\frac{2x^2+5x-10}{(x-1)(x+2)} &\equiv A+\frac{B}{x-1}+\frac{C}{x+2} \\ &\equiv \frac{A(x-1)(x+2)+B(x+2)+C(x-1)}{x-1}+\frac{C}{x+2}\end{aligned}$$

and so

$$2x^2+5x-10\equiv A(x-1)(x+2)+B(x+2)+C(x-1).$$

$\underline{x=1}$: $-3=3B\Rightarrow B=-1$.
$\underline{x=-2}$: $-12=-3C\Rightarrow C=4$.

$\underline{x = 0}$: $-10 = -2A + 2B - C \Rightarrow A = 2$.
Hence
$$\frac{2x^2 + 5x - 10}{(x-1)(x+2)} \equiv \underline{\underline{2 - \frac{1}{x-1} + \frac{4}{x+2}}}.$$

(b) Hence, or otherwise, expand $\frac{2x^2 + 5x - 10}{(x-1)(x+2)}$ in ascending powers of x, up to and including the term in x^2. Give each coefficient as a simplified fraction. (7)

Solution

$$\begin{aligned} -\frac{1}{x-1} &= (1-x)^{-1} \\ &= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots \\ &= 1 + x + x^2 + \ldots, \end{aligned}$$

$$\begin{aligned} \frac{1}{2+x} &= (2+x)^{-1} \\ &= 2^{-1} + (-1)2^{-2}x + \frac{(-1)(-2)}{2!}2^{-3}x^2 \\ &= \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 + \ldots, \end{aligned}$$

and

$$\begin{aligned} &\frac{2x^2 + 5x - 10}{(x-1)(x+2)} \\ \equiv\ & 2 - \frac{1}{x-1} + \frac{4}{x+2} \\ \equiv\ & 2 + (1 + x + x^2 + \ldots) + 4(\tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{1}{8}x^2 + \ldots) \\ \equiv\ & \underline{\underline{5 + \tfrac{3}{2}x^2 + \ldots}} \end{aligned}$$

12. (a) Use the binomial theorem to expand (5)

$$(3 - 2x)^{-2},\ |x| < \tfrac{2}{3},$$

in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

Solution

$$\begin{aligned}(2-3x)^{-2} &= [2+(-3x)]^{-2} \\ &= 2^{-2} + (-2)(2^{-3})(-3x) + \frac{(-2)(-3)}{2!}(2^{-4})(-3x)^2 \\ &\qquad + \frac{(-2)(-3)(-4)}{3!}(2^{-5})(-3x)^3 + \ldots \\ &= \underline{\underline{\tfrac{1}{4} + \tfrac{3}{4}x + \tfrac{27}{16}x^2 + \tfrac{27}{8}x^3 + \ldots}}\end{aligned}$$

$$\mathrm{f}(x) = \frac{a+bx}{(3-2x)^2},\ |x| < \tfrac{2}{3},\ \text{where } a \text{ and } b \text{ are constants.}$$

In the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, the coefficient of x is 0 and the coefficient of x^2 is $\frac{9}{16}$. Find

(b) the value of a and the value of b, (5)

Solution

	$\frac{1}{4}$	$+\frac{3}{4}x$	$+\frac{27}{16}x^2$	$+\frac{27}{8}x^3$
a	$\frac{1}{4}a$	$+\frac{3}{4}ax$	$+\frac{27}{16}ax^2$	$+\frac{27}{8}ax^3$
$+bx$	$+\frac{1}{4}bx$	$+\frac{3}{4}bx^2$	$+\frac{27}{16}bx^3$	$\ldots$

Hence

$$\frac{a+bx}{(3-2x)^2} = \tfrac{1}{4}a + (\tfrac{3}{4}a + \tfrac{1}{4}b)x + (\tfrac{27}{16}a + \tfrac{3}{4}b)x^2 + (\tfrac{27}{8}a + \tfrac{27}{16}b)x^3 + \ldots$$

Now,

$$\begin{aligned}\text{the coefficient of } x \text{ is } 0 &\Rightarrow \tfrac{3}{4}a + \tfrac{1}{4}b = 0 \\ &\Rightarrow b = -3a \\ &\Rightarrow \tfrac{27}{16}a + \tfrac{3}{4}(-3a) = \tfrac{9}{16} \\ &\Rightarrow -\tfrac{9}{16}a = \tfrac{9}{16} \\ &\Rightarrow \underline{\underline{a = -1}} \\ &\Rightarrow \underline{\underline{b = 3}}\end{aligned}$$

(c) the coefficient of x^3, giving your answer as a simplified fraction. (3)

Solution

The coefficient of x^3 is
$$\tfrac{27}{8} \times (-1) + \tfrac{27}{16} \times 3 = \underline{\underline{\tfrac{27}{16}}}.$$

13. (6)

$$f(x) = \frac{1}{\sqrt{9 + 4x^2}},\ |x| < \tfrac{3}{2}.$$

Find the first three non-zero terms of the binomial expansion of $f(x)$, in ascending powers of x. Give each coefficient as a simplified fraction.

Solution

$$\begin{aligned}\frac{1}{\sqrt{9+4x^2}} &= (9 + 4x^2)^{-\frac{1}{2}} \\ &= 9^{-\frac{1}{2}} + (-\tfrac{1}{2})(9^{-\frac{3}{2}})(4x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(9^{-\frac{5}{2}})(4x^2)^2 + \ldots \\ &= \underline{\underline{\tfrac{1}{3} - \tfrac{2}{27}x^2 + \tfrac{2}{81}x^4 + \ldots}}\end{aligned}$$

14. (a) Expand (5)

$$\frac{1}{(2 - 5x)^2},\ |x| < \tfrac{2}{5}$$

in ascending powers of x, up to and including the term in x^2, giving each term as a simplified fraction.

Solution

$$\begin{aligned}\frac{1}{(2-5x)^2} &= [2 + (-5x)]^{-2} \\ &= 2^{-2} + (-2)(2^{-3})(-5x) + \frac{(-2)(-3)}{2!}(2^{-4})(-5x)^2 + \ldots \\ &= \underline{\underline{\tfrac{1}{4} + \tfrac{5}{4}x + \tfrac{75}{16}x^2 + \ldots}}\end{aligned}$$

Given that the binomial expansion of $\dfrac{2+kx}{(2-5x)^2}$, $|x| < \frac{2}{5}$ is

$$\tfrac{1}{2} + \tfrac{7}{4}x + Ax^2 + \dots$$

(b) find the value of the constant k, (2)

Solution

	$\frac{1}{4}$	$+\frac{5}{4}x$	$+\frac{75}{16}x^2$
2	$\frac{1}{2}$	$+\frac{5}{2}x$	$+\frac{75}{8}x^2$
$+kx$	$\frac{1}{4}kx$	$+\frac{5}{4}kx^2$	$\dots$

Hence

$$\frac{2+kx}{(2-5x)^2} = \tfrac{1}{2} + (\tfrac{5}{2} + \tfrac{1}{4}k)x + (\tfrac{75}{16} + \tfrac{5}{4}k)x^2 + \dots$$

and

$$\tfrac{5}{2} + \tfrac{1}{4}k = \tfrac{7}{4} \Rightarrow \tfrac{1}{4}k = -\tfrac{3}{4} \Rightarrow \underline{\underline{k = -3}}.$$

(c) find the value of the constant A. (2)

Solution

$$A = \tfrac{75}{8} + \tfrac{5}{4} \times (-3) = \underline{\underline{\tfrac{45}{8}}}.$$

15.

$$\mathrm{f}(x) = \frac{6}{\sqrt{9-4x}}, \; |x| < \tfrac{9}{4}.$$

(a) Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction. (6)

Solution

$$
\begin{aligned}
\frac{6}{\sqrt{9-4x}} &= 6\left[9+(-4x)\right]^{-\frac{1}{2}} \\
&= 6\left[9^{-\frac{1}{2}} + (-\tfrac{1}{2})(9^{-\frac{3}{2}})(-4x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(9^{-\frac{5}{2}})(-4x)^2\right. \\
&\qquad \left. + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(9^{-\frac{7}{2}})(-4x)^3 + \ldots\right] \\
&= \underline{\underline{2 + \tfrac{4}{9}x + \tfrac{4}{27}x^2 + \tfrac{40}{729}x^3 + \ldots}}
\end{aligned}
$$

Use your answer to part (a) to find the binomial expansion in ascending powers of x, up to and including the term in x^3, of

(b) $g(x) = \dfrac{6}{\sqrt{9+4x}}$, $|x| < \frac{9}{4}$, (1)

Solution

$$g(x) = \underline{\underline{2 - \tfrac{4}{9}x + \tfrac{4}{27}x^2 - \tfrac{40}{729}x^3 + \ldots}}.$$

(c) $h(x) = \dfrac{6}{\sqrt{9-8x}}$, $|x| < \frac{9}{8}$. (2)

Solution

$$
\begin{aligned}
h(x) &= 2 + \tfrac{4}{9}(2x) + \tfrac{4}{27}(2x)^2 + \tfrac{40}{729}(2x)^3 + \ldots \\
&= \underline{\underline{2 + \tfrac{8}{9}x + \tfrac{16}{27}x^2 + \tfrac{320}{729}x^3 + \ldots}}
\end{aligned}
$$

16. Given (5)

$$f(x) = (2+3x)^{-3},\ |x| < \tfrac{2}{3},$$

find the binomial expansion of $f(x)$, in ascending powers of x, up to and including the term in x^3.

Give each coefficient as a simplified fraction.

Solution

$$
\begin{aligned}
(2+3x)^{-3} &= 2^{-3} + (-3)(2^{-4})(3x) + \frac{(-3)(-4)}{2!}(2^{-5})(3x)^2 \\
&\qquad + \frac{(-3)(-4)(-5)}{3!}(2^{-6})(3x)^3 + \dots \\
&= \underline{\underline{\tfrac{1}{8} - \tfrac{9}{16}x + \tfrac{27}{16}x^2 - \tfrac{135}{32}x^3 + \dots}}
\end{aligned}
$$

17. (a) Use the binomial expansion to show that (6)

$$\sqrt{\frac{1+x}{1-x}} \approx 1 + x + \tfrac{1}{2}x^2,\ |x| < 1.$$

Solution

$$
\begin{aligned}
\sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\
&= 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \dots \\
&= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \dots,
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\frac{1}{1-x}} &= [1 + (-x)]^{-\frac{1}{2}} \\
&= 1 + (-\tfrac{1}{2})(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \dots \\
&= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \dots,
\end{aligned}
$$

	1	$+\frac{1}{2}x$	$-\frac{1}{8}x^2$
1	1	$+\frac{1}{2}x$	$-\frac{1}{8}x^2$
$+\frac{1}{2}x$	$+\frac{1}{2}x$	$+\frac{1}{4}x^2$	$\dots$
$+\frac{3}{8}x^2$	$+\frac{3}{8}x^2$	$\dots$	$\dots$

and so

$$\sqrt{\frac{1+x}{1-x}} \approx \underline{\underline{1 + x + \tfrac{1}{2}x^2}}$$

(b) Substitute $x = \frac{1}{26}$ into (3)

$$\sqrt{\frac{1+x}{1-x}} = 1 + x + \tfrac{1}{2}x^2$$

to obtain an approximation to $\sqrt{3}$.

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

Solution

$$\begin{aligned}
&\sqrt{\frac{1+\frac{1}{26}}{1-\frac{1}{26}}} \approx 1 + \tfrac{1}{26} + \tfrac{1}{2}(\tfrac{1}{26})^2 \\
\Rightarrow\quad &\sqrt{\frac{27}{25}} \approx \tfrac{1405}{1352} \\
\Rightarrow\quad &\tfrac{3}{5}\sqrt{3} \approx \tfrac{1405}{1352} \\
\Rightarrow\quad &\sqrt{3} \approx \underline{\underline{\tfrac{7025}{4056}}}.
\end{aligned}$$

18. (a) Find the binomial expansion of (6)

$$\sqrt[3]{8-9x},\ |x| < \tfrac{8}{9},$$

in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

Solution

$$\begin{aligned}
\sqrt[3]{8-9x} &= [8 + (-9x)]^{\frac{1}{3}} \\
&= 8^{\frac{1}{3}} + \tfrac{1}{3}(8^{-\frac{2}{3}})(-9x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(8^{-\frac{5}{3}})(-9x)^2 \\
&\qquad + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(8^{-\frac{8}{3}})(-9x)^3 + \dots \\
&= \underline{\underline{2 - \tfrac{3}{4}x - \tfrac{9}{32}x^2 - \tfrac{45}{256}x^3 + \dots}}
\end{aligned}$$

(b) Use your expansion to estimate an approximate value for $\sqrt[3]{7100}$, giving your answer to 4 decimal places. State the value of x, which you use in your expansion, and show all your working. (3)

Solution

$$\sqrt[3]{7100} = 10\sqrt[3]{7.1}$$

so use $x = 0.1$:

$$\begin{aligned}
\sqrt[3]{7100} &= 10\sqrt[3]{7.1} \\
&= 10\sqrt[3]{8 - 9 \times 0.1} \\
&\approx 10\left[2 - \tfrac{3}{4}(0.1) - \tfrac{9}{32}(0.1)^2 - \tfrac{45}{256}(0.1)^3\right] \\
&= 10 \times 1.922\,011\,719 \text{ (FCD)} \\
&= 19.220\,119\,19 \text{ (FCD)} \\
&= \underline{\underline{19.220\,1 \text{ (4 dp)}}}.
\end{aligned}$$

19. Given that the binomial expansion of $(1 + kx)^{-4}$, $|kx| < 1$, is

$$1 - 6x + Ax^2 + \ldots,$$

(a) find the value of the constant k, (2)

Solution

$$\begin{aligned}
(1 + kx)^{-4} &= 1 + (-4)(kx) + \frac{(-4)(-5)}{2!}(kx)^2 + \ldots \\
&= 1 - 4kx + 10k^2x^2 + \ldots
\end{aligned}$$

Now,

$$-4k = -6 \Rightarrow \underline{\underline{k = \tfrac{3}{2}}}.$$

(b) find the value of the constant A, giving your answer in its simplest form. (3)

Solution

$$A = 10 \times (\tfrac{3}{2})^2 = \underline{\underline{\tfrac{45}{2}}}.$$

20. (a) Find the binomial expansion of (5)

$$\frac{1}{\sqrt{9 - 10x}},\ |x| < \tfrac{9}{10}$$

in ascending powers of x, up to and including the term in x^2. Give each coefficient as a simplified fraction.

Solution

$$\begin{aligned}\frac{1}{\sqrt{9-10x}} &= [9+(-10x)]^{-\frac{1}{2}} \\ &= 9^{-\frac{1}{2}} + (-\tfrac{1}{2})(9^{-\frac{3}{2}})(-10x) + \frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)}{2!}(9^{-\frac{5}{2}})(-10x)^2 + \dots \\ &= \underline{\underline{\tfrac{1}{3} + \tfrac{5}{27}x + \tfrac{25}{162}x^2 + \dots}}\end{aligned}$$

(b) Hence, or otherwise, find the expansion of (3)

$$\frac{3+x}{\sqrt{9-10x}},\ |x| < \tfrac{9}{10}$$

in ascending powers of x, up to and including the term in x^2. Give each coefficient as a simplified fraction.

Solution

	$\frac{1}{3}$	$+\frac{5}{27}x$	$+\frac{25}{162}x^2$
3	1	$+\frac{5}{9}x$	$+\frac{25}{54}x^2$
$+x$	$+\frac{1}{3}x$	$+\frac{5}{27}x^2$	$\dots$

and so

$$\frac{3+x}{\sqrt{9-10x}} = \underline{\underline{1 + \tfrac{8}{9}x + \tfrac{35}{54}x^2 + \dots}}$$

21. (a) Find the binomial series expansion of (5)

$$(4+5x)^{\frac{1}{2}}, |x| < \tfrac{4}{5},$$

in ascending powers of x, up to and including the term in x^2. Give each coefficient in its simplest form.

Solution

$$\begin{aligned}(4+5x)^{\frac{1}{2}} &= 4^{\frac{1}{2}} + \tfrac{1}{2}(4^{-\frac{1}{2}})(5x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4^{-\frac{3}{2}})(5x)^2 + \dots \\ &= \underline{\underline{2 + \tfrac{5}{4}x - \tfrac{25}{64}x^2 + \dots}}\end{aligned}$$

(b) Find the exact value of $(4+5x)^{\frac{1}{2}}$ when $x = \frac{1}{10}$. Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined. (1)

Solution

$$(4 + 5 \times \tfrac{1}{2})^{\frac{1}{2}} = \underline{\underline{\tfrac{3}{2}\sqrt{2}}}.$$

(c) Substitute $x = \frac{1}{10}$ into your binomial expansion from part (a) and hence find an approximate value for $\sqrt{2}$. Give your answer in the form $\frac{p}{q}$ where p and q are integers. (2)

Solution

Using the series in part (a) we get a value of $2\frac{31}{256} = \frac{543}{256}$. So

$$\tfrac{3}{2}\sqrt{2} \approx \tfrac{543}{256} \Rightarrow \underline{\underline{\sqrt{2} \approx \tfrac{181}{128}}}.$$

22. Use the binomial series to find the expansion of (6)

$$\frac{1}{(2+5x)^3},\ |x| < \tfrac{2}{5}$$

in ascending powers of x, up to and including the term in x^3.

Give each coefficient in its simplest form.

Solution

$$\begin{aligned}
\frac{1}{(2+5x)^3} &= (2+5x)^{-3} \\
&= \left[2(1+\tfrac{5x}{2})\right]^{-3} \\
&= \tfrac{1}{8}(1+\tfrac{5x}{2})^{-3} \\
&= \tfrac{1}{8}\left[1 + (-3)(\tfrac{5}{2}x) + \frac{(-3)(-4)}{2!}(\tfrac{5}{2}x)^2 + \frac{(-3)(-4)(-5)}{3!}(\tfrac{5}{2}x)^3 + \ldots\right] \\
&= \underline{\underline{\tfrac{1}{8} - \tfrac{15}{16}x + \tfrac{75}{16}x^2 - \tfrac{625}{32}x^3 + \ldots}}
\end{aligned}$$

23.

$$\mathrm{f}(x) = (2+kx)^{-3},\ |kx| < 2,\ \text{where } k \text{ is a positive constant.}$$

The binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^2 is

$$A + Bx + \tfrac{243}{16}x^2.$$

(a) Write down the value of A. (1)

Solution

$\underline{\underline{A = \tfrac{1}{8}}}$.

(b) Find the value of k. (3)

Solution

$$\begin{aligned}(2+3x)^{-3} &= 2^{-3} + (-3)(2^{-4})(kx) + \frac{(-3)(-4)}{2!}(2^{-5})(kx)^2 + \ldots \\ &= \tfrac{1}{8} - \tfrac{3}{16}kx + \tfrac{3}{16}k^2x^2 + \ldots\end{aligned}$$

and

$$\tfrac{3}{16}k^2 = \tfrac{243}{16} \Rightarrow \underline{\underline{k = 9}}.$$

(c) Write down the value of B. (2)

Solution

$B = -\tfrac{3}{16} \times 9 = \underline{\underline{-\tfrac{27}{16}}}$.