Dr Oliver Mathematics
Trigonometry: Multiple Angles

In this note, we will investigate the sine, cosine, and tangent of multiples angles.

We need the following:

$$\begin{aligned}
\sin(\alpha \pm \beta) &\equiv \sin\alpha\cos\beta \pm \sin\beta\cos\alpha, \\
\cos(\alpha \pm \beta) &\equiv \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \\
\tan(\alpha \pm \beta) &\equiv \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.
\end{aligned}$$

1 Sine, Cosine, and Tangent of $2A$

1.1 Sine of $2A$

$$\begin{aligned}
\sin 2A &\equiv \sin(A + A) \\
&\equiv \sin A\cos A + \sin A\cos A \\
&\equiv \underline{\underline{2\sin A\cos A}}.
\end{aligned}$$

1.2 Cosine of $2A$

$$\begin{aligned}
\cos 2A &\equiv \cos(A + A) \\
&\equiv \cos A\cos A - \sin A\sin A \\
&\equiv \underline{\underline{\cos^2 A - \sin^2 A}} \quad (*) \\
&\equiv \cos^2 A - (1 - \cos^2 A) \\
&\equiv \underline{\underline{2\cos^2 A - 1}}.
\end{aligned}$$

Returning to $(*)$,

$$\begin{aligned}
\cos 2A &\equiv \cos^2 A - \sin^2 A \\
&\equiv (1 - \sin^2 A) - \sin^2 A \\
&\equiv \underline{\underline{1 - 2\sin^2 A}}.
\end{aligned}$$

1.3 Tangent of $2A$

$$\begin{aligned}
\tan 2A &\equiv \tan(A + A) \\
&\equiv \frac{\tan A + \tan A}{1 - \tan A\tan A} \\
&\equiv \underline{\underline{\frac{2\tan A}{1 - \tan^2 A}}}.
\end{aligned}$$

1.4 Summary

$$\begin{aligned}
\sin 2A &\equiv 2\sin A\cos A;\\
\cos 2A &\equiv \cos^2 A - \sin^2 A,\\
&\equiv 2\cos^2 A - 1,\\
&\equiv 1 - 2\sin^2 A;\\
\tan 2A &\equiv \frac{2\tan A}{1-\tan^2 A}.
\end{aligned}$$

2 Sine, Cosine, and Tangent of $3A$

2.1 Sine of $3A$

$$\begin{aligned}
\sin 3A &\equiv \sin(2A + A)\\
&\equiv \sin 2A\cos A + \sin A\cos 2A\\
&\equiv (2\sin A\cos A)\cos A + \sin A(1-2\sin^2 A)\\
&\equiv 2\sin A\cos^2 A + \sin A(1-2\sin^2 A)\\
&\equiv 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A\\
&\equiv 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A\\
&\equiv \underline{\underline{3\sin A - 4\sin^3 A}}.
\end{aligned}$$

2.2 Cosine of $3A$

$$\begin{aligned}
\cos 3A &\equiv \cos(2A + A)\\
&\equiv \cos 2A\cos A - \sin 2A\sin A\\
&\equiv (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A\\
&\equiv 2\cos^3 A - \cos A - 2\sin^2 A\cos A\\
&\equiv 2\cos^3 A - \cos A - 2\cos A(1-\cos^2 A)\\
&\equiv 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A\\
&\equiv \underline{\underline{4\cos^3 A - 3\cos A}}.
\end{aligned}$$

2.3 Tangent of $3A$

$$\begin{aligned}
\tan 3A &\equiv \tan(2A + A)\\
&\equiv \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A}\\
&\equiv \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \left(\frac{2\tan A}{1-\tan^2 A}\right)\tan A}
\end{aligned}$$

multiply top and bottom by $(1-\tan^2 A)$:

$$\begin{aligned}
&\equiv \frac{2\tan A + \tan A(1-\tan^2 A)}{(1-\tan^2 A) - (2\tan A)\tan A} \\
&\equiv \frac{2\tan A + \tan A - \tan^3 A)}{1-\tan^2 A - 2\tan^2 A} \\
&\equiv \underline{\underline{\frac{3\tan A - \tan^3 A}{1-3\tan^2 A}}}.
\end{aligned}$$

2.4 Summary

$$\begin{aligned}
\sin 3A &\equiv 3\sin A - 4\sin^3 A; \\
\cos 3A &\equiv 4\cos^3 A - 3\cos A; \\
\tan 3A &\equiv \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}.
\end{aligned}$$

3 Sine, Cosine, and Tangent of $4A$

Shall we do

$$\sin 4A \equiv \sin(3A + A)$$

or do

$$\sin 4A \equiv \sin(2A + 2A)?$$

Either is possible but we will do

$$\text{trig } 4A \equiv \text{trig } (2A + 2A).$$

3.1 Sine of $4A$

There are a number of possible results.

$$\begin{aligned}
\sin 4A &\equiv \sin(2A + 2A) \\
&\equiv 2\sin 2A \cos 2A \\
&\equiv 2(2\sin A\cos A)(1-2\sin^2 A) \\
&\equiv 4\sin A\cos A(1-2\sin^2 A) \\
&\equiv \underline{\underline{4\sin A\cos A - 8\sin^3 A\cos A}},
\end{aligned}$$

or

$$\begin{aligned}
\sin 4A &\equiv \sin(2A + 2A) \\
&\equiv 2\sin 2A \cos 2A \\
&\equiv 2(2\sin A\cos A)(\cos^2 - \sin^2 A) \\
&\equiv 4\sin A\cos A(\cos^2 - \sin^2 A) \\
&\equiv \underline{\underline{4\sin A\cos^3 A - 4\sin^3 A\cos A}},
\end{aligned}$$

etc.

3.2 Cosine of $4A$

$$
\begin{aligned}
\cos 4A &\equiv \cos(2A + 2A) \\
&\equiv \cos^2 2A - \sin^2 2A \\
&\equiv (\cos 2A)^2 - (\sin 2A)^2 \\
&\equiv (2\cos^2 A - 1)^2 - (2\sin A \cos A)^2 \\
&\equiv (4\cos^4 A - 4\cos^2 A + 1) - 4\sin^2 A \cos^2 A \\
&\equiv 4\cos^4 A - 4\cos^2 A + 1 - 4\cos^2 A(1 - \cos^2 A) \\
&\equiv 4\cos^4 A - 4\cos^2 A + 1 - 4\cos^2 A + 4\cos^4 A \\
&\equiv \underline{\underline{8\cos^4 A - 8\cos^2 A + 1}}.
\end{aligned}
$$

3.3 Tangent of $4A$

$$
\begin{aligned}
\tan 4A &\equiv \tan(2A + 2A) \\
&\equiv \frac{\tan 2A + \tan 2A}{1 - \tan 2A \tan 2A} \\
&\equiv \frac{\frac{2\tan A}{1-\tan^2 A} + \frac{2\tan A}{1-\tan^2 A}}{1 - \left(\frac{2\tan A}{1-\tan^2 A}\right)\left(\frac{2\tan A}{1-\tan^2 A}\right)} \\
&\equiv \frac{\frac{4\tan A}{1-\tan^2 A}}{1 - \left(\frac{4\tan^2 A}{(1-\tan^2 A)^2}\right)}
\end{aligned}
$$

multiply top and bottom by $(1 - \tan^2 A)^2$:

$$
\begin{aligned}
&\equiv \frac{4\tan A}{(1 - \tan^2 A)^2 - 4\tan^2 A} \\
&\equiv \frac{4\tan A(1 - \tan^2 A)}{(1 - 2\tan^2 A + \tan^4 A) - 4\tan^2 A} \\
&\equiv \underline{\underline{\frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}}}.
\end{aligned}
$$

3.4 Summary

$$
\begin{aligned}
\sin 4A &\equiv 4\sin A \cos A - 8\sin^3 A \cos A \\
&\equiv 4\sin A \cos^3 A - 4\sin^3 A \cos A; \\
\cos 4A &\equiv 8\cos^4 A - 8\cos^2 A + 1; \\
\tan 4A &\equiv \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}.
\end{aligned}
$$

4 Alternative Approach

As you may know from your study of Further Mathematics,

$$\cos n\theta + \mathrm{i}\sin n\theta \equiv (\cos\theta + \mathrm{i}\sin\theta)^n,\ n \in \mathbb{Z},$$

where i is the square root of -1.

We will use that here.

5 Sine, Cosine, and Tangent of $2A$

5.1 $(\cos A + \mathrm{i}\sin A)^2$

$$\begin{aligned}\cos 2A + \mathrm{i}\sin 2A &\equiv (\cos A + \mathrm{i}\sin A)^2\\ &\equiv \binom{2}{0}(\cos A)^2 + \binom{2}{1}(\cos A)(\mathrm{i}\sin A) + \binom{2}{2}(\mathrm{i}\sin A)^2\\ &\equiv \cos^2 A + 2\,\mathrm{i}\sin A\cos A - \sin^2 A\\ &\equiv (\cos^2 A - \sin^2 A) + \mathrm{i}(2\sin A\cos A).\end{aligned}$$

5.2 Sine of $2A$

Take the imaginary part of the answer:

$$\sin 2A \equiv \underline{\underline{2\sin A\cos A}}.$$

5.3 Cosine of $2A$

Take the real part of the answer:

$$\cos 2A \equiv \underline{\underline{\cos^2 A - \sin^2 A}}.$$

5.4 Tangent of $2A$

$$\begin{aligned}\tan 2A &\equiv \frac{\operatorname{Im}\left[(\cos A + \mathrm{i}\sin A)^2\right]}{\operatorname{Re}\left[(\cos A + \mathrm{i}\sin A)^2\right]}\\ &\equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}\end{aligned}$$

multiply top and bottom by $\frac{1}{\cos^2 A}$:

$$\begin{aligned} &\equiv \frac{2\frac{\sin A}{\cos A}}{1 - \frac{\sin^2 A}{\cos^2 A}} \\ &\equiv \underline{\underline{\frac{2\tan A}{1 - \tan^2 A}}}. \end{aligned}$$

6 Sine, Cosine, and Tangent of $3A$

6.1 $(\cos A + \mathrm{i}\sin A)^3$

$$\begin{aligned} & \cos 3A + \mathrm{i}\sin 3A \\ \equiv\ & (\cos A + \mathrm{i}\sin A)^3 \\ \equiv\ & \binom{3}{0}(\cos A)^3 + \binom{3}{1}(\cos A)^2(\mathrm{i}\sin A) + \binom{3}{2}(\cos A)(\mathrm{i}\sin A)^2 + \binom{3}{3}(\mathrm{i}\sin A)^3 \\ \equiv\ & \cos^3 A + 3\mathrm{i}\cos^2 A\sin A - 3\cos A\sin^2 A - \mathrm{i}\sin^3 A \\ \equiv\ & (\cos^3 A - 3\cos A\sin^2 A) + \mathrm{i}(3\cos^2 A\sin A - \sin^3 A). \end{aligned}$$

6.2 Sine of $3A$

Take the imaginary part of the answer:

$$\begin{aligned} \sin 3A &\equiv 3\cos^2 A\sin A - \sin^3 A \\ &\equiv 3\sin A(1 - \sin^2 A) - \sin^3 A \\ &\equiv 3\sin A - 3\sin^2 A - \sin^3 A \\ &\equiv \underline{\underline{3\sin A - 4\sin^3 A}}. \end{aligned}$$

6.3 Cosine of $3A$

Take the real part of the answer:

$$\begin{aligned} \cos 3A &\equiv \cos^3 A - 3\cos A\sin^2 A \\ &\equiv \cos^3 A - 3\cos A(1 - \cos^2 A) \\ &\equiv \cos^3 A - 3\cos A + 3\cos^3 A \\ &\equiv \underline{\underline{4\cos^3 A - 3\cos A}}. \end{aligned}$$

6.4 Tangent of $3A$

$$
\begin{aligned}
\tan 3A &\equiv \frac{\operatorname{Im}\left[(\cos A + \mathrm{i}\sin A)^3\right]}{\operatorname{Re}\left[(\cos A + \mathrm{i}\sin A)^3\right]} \\
&\equiv \frac{3\cos^2 A \sin A - \sin^3 A}{\cos^3 A - 3\cos A \sin^2 A}
\end{aligned}
$$

multiply top and bottom by $\frac{1}{\cos^3 A}$:

$$
\begin{aligned}
&\equiv \frac{3\frac{\sin A}{\cos A} - \frac{\sin^3 A}{\cos^3 A}}{1 - 3\frac{\sin^2 A}{\cos^2 A}} \\
&\equiv \underline{\underline{\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}}.
\end{aligned}
$$

7 Sine, Cosine, and Tangent of $4A$

7.1 $(\cos A + \mathrm{i}\sin A)^4$

$$
\begin{aligned}
& \cos 4A + \mathrm{i}\sin 4A \\
\equiv\ & (\cos A + \mathrm{i}\sin A)^4 \\
\equiv\ & \binom{4}{0}(\cos A)^4 + \binom{4}{1}(\cos A)^3(\mathrm{i}\sin A) + \binom{4}{2}(\cos A)^2(\mathrm{i}\sin A)^2 \\
& \qquad + \binom{4}{3}(\cos A)(\mathrm{i}\sin A)^3 + \binom{4}{4}(\mathrm{i}\sin A)^4 \\
\equiv\ & \cos^4 A + 4\mathrm{i}\cos^3 A \sin A - 6\cos^2 A \sin^2 A - 4\mathrm{i}\cos A \sin^3 A + \sin^4 A \\
\equiv\ & (\cos^4 A - 6\cos^2 A \sin^2 A + \sin^4 A) + \mathrm{i}(4\cos^3 A \sin A - 4\cos A \sin^3 A).
\end{aligned}
$$

7.2 Sine of $4A$

Take the imaginary part of the answer:

$$\sin 4A \equiv \underline{\underline{4\cos^3 A \sin A - 4\cos A \sin^3 A}}.$$

7.3 Cosine of $4A$

Take the real part of the answer:

$$
\begin{aligned}
\cos 4A &\equiv \cos^4 A - 6\cos^2 A \sin^2 A + \sin^4 A \\
&\equiv \cos^4 A - 6\cos^2 A(1 - \cos^2 A) + (1 - \cos^2 A)^2 \\
&\equiv \cos^4 A - 6\cos^2 A + 6\cos^2 A + (1 - 2\cos^2 A + \cos^4 A) \\
&\equiv \cos^4 A - 6\cos^2 A + 6\cos^4 A + (1 - 2\cos^2 A + \cos^4 A) \\
&\equiv \underline{\underline{8\cos^4 A - 8\cos^2 A + 1}}.
\end{aligned}
$$

7.4 Tangent of $4A$

$$\begin{aligned}\tan 4A &\equiv \frac{\operatorname{Im}\left[(\cos A + \mathrm{i}\sin A)^4\right]}{\operatorname{Re}\left[(\cos A + \mathrm{i}\sin A)^4\right]} \\ &\equiv \frac{4\cos^3 A \sin A - 4\cos A \sin^3 A}{\cos^4 A - 6\cos^2 A \sin^2 A + \sin^4 A}\end{aligned}$$

multiply top and bottom by $\frac{1}{\cos^4 A}$:

$$\begin{aligned}&\equiv \frac{4\frac{\sin A}{\cos A} - 4\frac{\sin^3 A}{\cos^3 A}}{1 - 6\frac{\sin^2 A}{\cos^2 A} + \frac{\sin^4 A}{\cos^3 A}} \\ &\equiv \underline{\underline{\frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}}}.\end{aligned}$$

8 What I Prefer

Although both methods are correct, I prefer "Alternative Approach": not only do you get the complete expansions but you get the tangent formula in a couple of lines.