Square root of 2

Dr Oliver

Dr Oliver Mathematics

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$\sqrt{2}$ is an irrational real number

Suppose that $\sqrt{2}$ is rational. Then there exist integers p and q, in lowest terms, such that

$$\sqrt{2} = \frac{p}{q}.$$

Then

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2.$$

So p^2 is even and hence p is even. So there in an integer k such that $p = 2k$. Then

$$p^2 = 2q^2 \Rightarrow (2k)^2 = 2q^2 \Rightarrow 4k^2 = 2q^2 \Rightarrow q^2 = 2k^2.$$

Hence q^2 is an even and so q is even. But this contradicts the assumption that p and q have no common factor other than 1.

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$\sqrt{-1}$ is an irrational real number

Suppose that $\sqrt{-1}$ is rational.

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$\sqrt{-1}$ is an irrational real number

Suppose that $\sqrt{-1}$ is rational. Then there exist integers p and q, with $q \neq 0$, such that

$$\sqrt{-1} = \frac{p}{q}.$$

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$\sqrt{-1}$ is an irrational real number

Suppose that $\sqrt{-1}$ is rational. Then there exist integers p and q, with $q \neq 0$, such that

$$\sqrt{-1} = \frac{p}{q}.$$

Then

$$\sqrt{-1} = \frac{p}{q} \Rightarrow -1 = \frac{p^2}{q^2} \Rightarrow p^2 = -q^2.$$

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$\sqrt{-1}$ is an irrational real number

Suppose that $\sqrt{-1}$ is rational. Then there exist integers p and q, with $q \neq 0$, such that

$$\sqrt{-1} = \frac{p}{q}.$$

Then

$$\sqrt{-1} = \frac{p}{q} \Rightarrow -1 = \frac{p^2}{q^2} \Rightarrow p^2 = -q^2.$$

Now p is an integer and so $p^2 \geqslant 0$ and q was an non-zero integer and so $-q^2 < 0$, giving us our logical contradiction.

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The issue

The conclusion that

$$\text{“}\sqrt{2}\text{ is an irrational number”}$$

is wrong.

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The issue

The conclusion that

$$\text{“}\sqrt{2}\text{ is an irrational number”}$$

is wrong. The conclusion that

$$\text{“}\sqrt{2}\text{ is not a rational number”}$$

is correct.

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The issue

The conclusion that

$$\text{“}\sqrt{2}\text{ is an irrational number”}$$

is wrong. The conclusion that

$$\text{“}\sqrt{2}\text{ is not a rational number”}$$

is correct. Indeed, at this point, we do not even have a logical basis for claiming that $\sqrt{2}$ is a number.

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Natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

The set $\mathbb{N}$ is closed under the operations of addition and multiplication. But it is not closed under subtraction: whatever the result of

$$2 - 6$$

is, it is *not* a number.

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Integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

The set $\mathbb{Z}$ is closed under the operations of addition, subtraction and multiplication. But it is not closed under addition: whatever the result of

$$3 \div 7$$

is, it is *not* a number.

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Rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$$

The set $\mathbb{Q}$ is closed under the operations of addition, subtraction, multiplication, and division. But it is not closed under the extraction of roots: whatever the result of

$$\sqrt{2}$$

is, it is *not* a number.

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Real numbers

$\sqrt{2}$ does not become a number until we expand our set or rational numbers to include it.

$$\mathbb{R} = \{?\}\,.$$

The only trouble is that we do not ever give anything even approximating a definition of what the real numbers are.

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What are the real numbers?

Three serious analytic approaches

- using Cauchy sequences of rational numbers to generate $\mathbb{R}$ as the completion of $\mathbb{Q}$
- Dedekind cut
- Stevin decimals

and one piece of waffle

- a point on a number line

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Is $\sqrt{2}$ equal to?

1.414 213 562 373 095 048 801 688 724 209 698 078 569 671 875
376 948 073 176 679 737 990 732 478 462 107 038 850 387 534
327 641 572 735 013 846 230 912 297 024 924 836 055 850 737
212 644 121 497 099 935 831 413 222 665 927 505 592 755 799
950 501 152 782 060 571 470 109 559 971 605 970 274 534 596
862 014 728 517 418 640 889 198 609 552 329 230 484 308 714
321 450 839 762 603 627 995 251 407 989 687 253 396 546 331
808 829 640 620 615 258 352 395 054 745 750 287 759 961 729
835 575 220 337 531 857 011 354 374 603 408 498 847 160 386
899 970 699 004 815 030 544 027 790 316 454 247 823 068 492
936 918 621 580 578 463 111 596 668 713 013 015 618 568 987
237 235 288 509 264 861 249 497 715 421 833 420 428 568 606
014 682 472 077 143 585 487 415 565 706 967 765 372 022 648
544 701 585 880 162 075 847 492 265 722 600 208 558 446 652
145 839 889 394 437 092 659 180 031 138 824 646 815 708 . . .

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