Dr Oliver Mathematics
The 'Differentiate-Integrate' (DI) Method

Example 1
Find

$$\int x^2 \mathrm{e}^{3x} \, \mathrm{d}x.$$

Solution 1
We will use integration by parts twice.

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x} \Rightarrow v = \tfrac{1}{3}\mathrm{e}^{3x}$$

Now,

$$\int x^2 \mathrm{e}^{3x} \, \mathrm{d}x = \tfrac{1}{3}x^2\mathrm{e}^{3x} - \int \tfrac{2}{3}x\mathrm{e}^{3x} \, \mathrm{d}x$$

$$u = \tfrac{2}{3}x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \tfrac{2}{3}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x} \Rightarrow v = \tfrac{1}{3}\mathrm{e}^{3x}$$

$$\begin{aligned} &= \tfrac{1}{3}x^2\mathrm{e}^{3x} - \left[\tfrac{2}{9}x\mathrm{e}^{3x} - \int \tfrac{2}{9}\mathrm{e}^{3x} \, \mathrm{d}x\right] \\ &= \tfrac{1}{3}x^2\mathrm{e}^{3x} - \left[\tfrac{2}{9}x\mathrm{e}^{3x} - \tfrac{2}{27}\mathrm{e}^{3x}\right] + c \\ &= \underline{\underline{\tfrac{1}{3}x^2\mathrm{e}^{3x} - \tfrac{2}{9}x\mathrm{e}^{3x} + \tfrac{2}{27}\mathrm{e}^{3x} + c.}} \end{aligned}$$

You may have decided to do

$$\tfrac{2}{3}\int x\mathrm{e}^{3x} \, \mathrm{d}x, \; u = x, \; \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x},$$

and that is fine.

Or, we could do …

Solution 1

	Differentiate	Integrate
$+$	x^2	e^{3x}
$-$	$2x$	$\frac{1}{3}\mathrm{e}^{3x}$
$+$	2	$\frac{1}{9}\mathrm{e}^{3x}$
$-$	0	$\frac{1}{27}\mathrm{e}^{3x}$

$$\int x^2 \mathrm{e}^{3x}\,\mathrm{d}x = \underline{\underline{\tfrac{1}{3}x^2\mathrm{e}^{3x} - \tfrac{2}{9}x\mathrm{e}^{3x} + \tfrac{2}{27}\mathrm{e}^{3x} + c}},$$

where we indicate in orange ($\frac{1}{3}x^2\mathrm{e}^{3x}$), red ($-\frac{2}{9}x\mathrm{e}^{3x}$), and yellow ($+\frac{2}{27}\mathrm{e}^{3x}$) respectively.

How does this work?

	Differentiate	Integrate
$+$	u	$\frac{\mathrm{d}v}{\mathrm{d}x}$
$-$	$\frac{\mathrm{d}u}{\mathrm{d}x}$	v
$+$	$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$	$\int v\,\mathrm{d}x$

Now,

$$\begin{aligned}\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \\ &= uv - \left[\frac{\mathrm{d}u}{\mathrm{d}x}\left(\int v\,\mathrm{d}x\right) - \int \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\left(\int v\,\mathrm{d}x\right)\,\mathrm{d}x\right] \\ &= uv - \frac{\mathrm{d}u}{\mathrm{d}x}\left(\int v\,\mathrm{d}x\right) + \int \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\left(\int v\,\mathrm{d}x\right)\,\mathrm{d}x.\end{aligned}$$

Now,

- we need an alternating sign of $+, -, +, \ldots,$
- then you write down the orange boxes,
- then you write down the red boxes,
- then you write down the yellow boxes, etc, until you are finished.
- This method means you need (marginally) less writing down.

Example 2
Find
$$\int x \sin x \, dx.$$

Solution 2

	Differentiate	Integrate
$+$	x	$\sin x$
$-$	1	$-\cos x$
$+$	0	$-\sin x$

$$\begin{aligned}\int x \sin x \, dx &= (x)(-\cos x) - (1)(-\sin x) + c \\ &= \underline{\underline{-x \cos x + \sin x + c}}.\end{aligned}$$

Example 3
Find
$$\int 12x^2(3+2x)^5 \, dx.$$

Solution 3

	Differentiate	Integrate
$+$	$12x^2$	$(3+2x)^5$
$-$	$24x$	$\frac{1}{12}(3+2x)^6$
$+$	24	$\frac{1}{168}(3+2x)^7$
$-$	0	$\frac{1}{1\,344}(3+2x)^8$

$$\begin{aligned}\int 12x^2(3+2x)^5 \, dx &= (12x^2)\left[\tfrac{1}{12}(3+2x)^6\right] - (24x)\left[\tfrac{1}{168}(3+2x)^8\right] \\ &\qquad + (24)\left[\tfrac{1}{2\,688}(3+2x)^8\right] + c \\ &= \underline{\underline{x^2(3+2x)^6 - \tfrac{1}{7}(3+2x)^8 + \tfrac{1}{112}(3+2x)^8 + c}}.\end{aligned}$$

Example 4
Find

$$\int 2x^2 \sec^2 x \tan x \, \mathrm{d}x.$$

Solution 4

	Differentiate	Integrate
$+$	x^2	$2\sec^2 x \tan x = 2(\sec x)(\sec x \tan x)$
$-$	$2x$	$\sec^2 x$
$+$	2	$\tan x$
$-$	0	$\ln\lvert\sec x\rvert$

$$\begin{aligned}\int 2x^2 \sec^2 x \tan x \, \mathrm{d}x &= \int (x^2)(2\sec^2 x \tan x) \, \mathrm{d}x \\ &= \underline{\underline{x^2 \sec^2 x - 2x \tan x + 2\ln\lvert\sec x\rvert + c}}.\end{aligned}$$

If $u = ax^n$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ is of the form $\sin bx$, e^{cx}, etc, certainly you will get down to zero in the differentiate column.

But what if you don't get there? What if your choice of u and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ ensures you don't get zero: what then?

Example 5
Find

$$\int \arcsin x \, \mathrm{d}x.$$

Solution 5

	Differentiate	Integrate
$+$	$\arcsin x$	1
$-$	$\dfrac{1}{\sqrt{1-x^2}}$	x

$$
\begin{aligned}
\int \arcsin x \,\mathrm{d}x &= \int (\arcsin x)(1)\,\mathrm{d}x \\
&= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= x \arcsin x - (-\sqrt{1-x^2}) + c \\
&= \underline{\underline{x \arcsin x + \sqrt{1-x^2} + c}}.
\end{aligned}
$$

Or if we simply go round in circles?

Example 6
Find

$$\int \mathrm{e}^{2x} \sin 3x \,\mathrm{d}x.$$

Solution 6

	Differentiate	Integrate
$+$	e^{2x}	$\sin 3x$
$-$	$2\mathrm{e}^{2x}$	$-\frac{1}{3}\cos 3x$
$+$	$4\mathrm{e}^{2x}$	$-\frac{1}{9}\sin 3x$

$$
\begin{aligned}
& \int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x = (\mathrm{e}^{2x})(-\tfrac{1}{3}\cos 3x) - (2\mathrm{e}^{2x})(-\tfrac{1}{9}\sin 3x) + \int (4\mathrm{e}^{2x})(-\tfrac{1}{9}\sin 3x)\,\mathrm{d}x \\
\Rightarrow\ & \int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x = -\tfrac{1}{3}\cos 3x + \tfrac{2}{9}\mathrm{e}^{2x}\sin 3x - \tfrac{4}{9}\int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x \\
\Rightarrow\ & \tfrac{13}{9}\int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x = -\tfrac{1}{3}\mathrm{e}^{2x}\cos 3x + \tfrac{2}{9}\mathrm{e}^{2x}\sin 3x \\
\Rightarrow\ & \tfrac{13}{9}\int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x = \tfrac{1}{9}\mathrm{e}^{2x}(2\sin 3x - 3\cos 3x) \\
\Rightarrow\ & \int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x = \underline{\underline{\tfrac{1}{13}\mathrm{e}^{2x}(2\sin 3x - 3\cos 3x) + c}}.
\end{aligned}
$$

The next time you integrate by parts, especially two or more times, think back to we you will have learned.

Do you like Differentiate-Integrate method? If you do, great! If not, that's okay — but it is certainly one for the mathematical toolbox that you carry.