Dr Oliver Mathematics
Mathematics
Binomial Expansion
Past Examination Questions

This booklet consists of 24 questions across a variety of examination topics.
The total number of marks available is 128.

For $n \in \mathbb{N}$,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n.$$

1. Find the first three terms, in ascending powers of x, of the binomial expansion of $(3+2x)^5$, giving each term in its simplest form. (4)

Solution

$$\begin{aligned}(3+2x)^5 &= 3^5 + \binom{5}{1} 3^4 (2x)^1 + \binom{5}{2} 3^3 (2x)^2 + \ldots \\ &= \underline{\underline{243 + 810x + 1080x^2 + \ldots}}\end{aligned}$$

2. (a) Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1+px)^{12}$, where p is a non-zero constant. (2)

Solution

$$\begin{aligned}(1+px)^{12} &= 1 + \binom{12}{1} (px)^1 + \binom{12}{2} (px)^2 + \ldots \\ &= \underline{\underline{1 + 12px + 66p^2x^2 + \ldots}}\end{aligned}$$

Given that, in the expansion of $(1+px)^{12}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $11q$,

(b) find the value of p and the find the value of q. (4)

Solution

Well, we have

$$12p = -q \text{ and } 66p^2 = 11q$$

and

$$\begin{aligned} 66p^2 = 11(-12p) &\Rightarrow 66p^2 = -132p \\ &\Rightarrow 66p^2 + 132p = 0 \\ &\Rightarrow 66p(p+2) = 0 \\ &\Rightarrow \underline{\underline{p = -2}} \text{ (as } p \neq 0) \\ &\Rightarrow \underline{\underline{q = 24}}. \end{aligned}$$

3. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of (2)

$$(1+px)^9,$$

where p is a constant.

Solution

$$\begin{aligned} (1+px)^9 &= 1 + \binom{9}{1}(px)^1 + \binom{9}{2}(px)^2 + \ldots \\ &= \underline{\underline{1 + 9px + 36p^2x^2 + \ldots}} \end{aligned}$$

The first 3 terms are 1, $36x$, and qx^2, where q is a constant.

(b) Find the values of p and the the values of q. (4)

Solution

$$9p = 36 \Rightarrow \underline{\underline{p = 4}}$$

and

$$q = 36p^2 = 36 \times 4^2 = \underline{\underline{576}}.$$

4. Find the first three terms, in ascending powers of x, of the binomial expansion of $(2+x)^6$, giving each term in its simplest form. (4)

Solution

$$\begin{aligned}(2+x)^6 &= 2^6 + \binom{6}{1}2^5x^1 + \binom{6}{2}2^4x^2 + \dots \\ &= \underline{\underline{64 + 192x + 240x^2 + \dots}}\end{aligned}$$

5. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1-2x)^5$. Give each term in its simplest form. (2)

Solution

$$\begin{aligned}(1-2x)^5 &= [1+(-2x)]^5 \\ &= 1 + \binom{5}{1}(-2x)^1 + \binom{5}{2}(-2x)^2 + \binom{5}{3}(-2x)^3 \dots \\ &= \underline{\underline{1 - 10x + 40x^2 - 80x^3 + \dots}}\end{aligned}$$

(b) If x is small, so that x^2 and higher powers can be ignored, show that (4)

$$(1+x)(1-2x)^5 \approx 1 - 9x.$$

Solution

	1	$-10x$
1	1	$-10x$
$+x$	$+x$	$\dots$

So

$$(1+x)(1-2x)^5 \approx 1 - 10x + x = \underline{\underline{1 - 9x}}.$$

6. (a) Find the first four terms, in ascending powers of x, of the binomial expansion of (3)

$$(1+kx)^6,$$

where k is a non-zero constant.

Solution

$$(1+kx)^6 = 1 + \binom{6}{1}(kx)^1 + \binom{6}{2}(kx)^2 + \binom{6}{3}(kx)^3 + \dots$$
$$= \underline{\underline{1 + 6kx + 15k^2x^2 + 20k^3x^3 + \dots}}$$

Given that, in this expansion, the coefficients of x and x^2 are equal,

(b) the value of k, (2)

Solution

$$\begin{aligned} 6k = 15k^2 &\Rightarrow 15k^2 - 6k = 0 \\ &\Rightarrow 3k(5k-2) = 0 \\ &\Rightarrow 5k - 2 = 0 \text{ (as } k \neq 0) \\ &\Rightarrow \underline{\underline{k = \tfrac{2}{5}}}. \end{aligned}$$

(c) the coefficient of x^3. (1)

Solution

$$20k^3 = 20 \times (\tfrac{2}{5})^3 = \underline{\underline{\tfrac{32}{25}}}.$$

7. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + \frac{1}{2}x)^{10}$, giving each term in its simplest form. (4)

Solution

$$(1 + \tfrac{1}{2}x)^{10} = 1 + \binom{10}{1}(\tfrac{1}{2}x)^1 + \binom{10}{2}(\tfrac{1}{2}x)^2 + \binom{10}{3}(\tfrac{1}{2}x)^3 + \dots$$
$$= \underline{\underline{1 + 5x + \tfrac{45}{4}x^2 + 15x^3 + \dots}}$$

(b) Use your expansion to estimate the value of $(1.005)^{10}$, giving your answer to 5 decimal places. (3)

Solution

$$\begin{aligned}(1.005)^{10} &= (1 + \tfrac{1}{2} \times 0.01)^{10} \\ &\approx 1 + 5(0.01) + \tfrac{45}{4}(0.01)^2 + 15(0.01)^3 \\ &= 1.051\,137 \\ &= \underline{\underline{1.051\,14 \text{ (5 dp)}}}.\end{aligned}$$

8. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + ax)^{10}$, where a is a non-zero constant. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(1 + ax)^{10} &= 1 + \binom{10}{1}(ax)^1 + \binom{10}{2}(ax)^2 + \binom{10}{3}(ax)^3 + \dots \\ &= \underline{\underline{1 + 10ax + 45a^2x^2 + 120a^3x^3 + \dots}}\end{aligned}$$

Given that, in this expansion, the coefficient of x^3 is double the coefficient of x^2,

(b) find the value of a. (2)

Solution

$$\begin{aligned}120a^3 = 90a^2 &\Rightarrow 120a^3 - 90a^2 = 0 \\ &\Rightarrow 30a^2(4a - 3) = 0 \\ &\Rightarrow \underline{\underline{a = \tfrac{3}{4}}} \text{ (as } a \neq 0)\end{aligned}$$

9. Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(3 - 2x)^5$, giving each term in its simplest form. (4)

Solution

$$\begin{aligned}(3-2x)^5 &= [3+(-2x)]^5 \\ &= 3^5 + \binom{5}{1}3^4(-2x)^1 + \binom{5}{2}3^3(-2x)^2 + \ldots \\ &= \underline{\underline{243 - 810x + 1080x^2 + \ldots}}\end{aligned}$$

10. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2+kx)^7$, where k is a constant. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(2+kx)^7 &= 2^7 + \binom{7}{1}2^6(kx)^1 + \binom{7}{2}2^5(kx)^2 + \ldots \\ &= \underline{\underline{128 + 448kx + 672k^2x^2 + \ldots}}\end{aligned}$$

Given that, in this expansion, the coefficient of x^2 is 6 times the coefficient of x,

(b) find the value of k. (2)

Solution

$$\begin{aligned}672k^2 = 2688k &\Rightarrow 672k^2 - 2688k = 0 \\ &\Rightarrow 672k(k-4) = 0 \\ &\Rightarrow \underline{\underline{k=0}} \text{ or } \underline{\underline{k=4}}.\end{aligned}$$

11. Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(3-x)^6,$$

and simply each term.

Solution

$$\begin{aligned}(3-x)^6 &= [3+(-x)]^6 \\ &= 3^6 + \binom{6}{1}3^5(-x)^1 + \binom{6}{2}3^4(-x)^2 + \ldots \\ &= \underline{\underline{729 - 1458x + 1215x^2 + \ldots}}\end{aligned}$$

12. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1+ax)^7$, where a is a constant. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(1+ax)^7 &= 1 + \binom{7}{1}(ax)^1 + \binom{7}{2}(ax)^2 + \binom{7}{3}(ax)^3 + \ldots \\ &= \underline{\underline{1 + 7ax + 21a^2x^2 + 35a^3x^3 + \ldots}}\end{aligned}$$

Given that the coefficient of x^2 is 525,

(b) find the possible values of a. (2)

Solution

$$\begin{aligned}21a^2 = 525 \Rightarrow a^2 &= 25 \\ &\Rightarrow \underline{\underline{a=-5}} \text{ or } \underline{\underline{a=5}}.\end{aligned}$$

13. Given that $\binom{40}{4} = \dfrac{40!}{4!b!}$,

(a) write down the value of b. (1)

Solution

$\underline{\underline{b = 36}}$.

In the binomial expansion of $(1+x)^{40}$, the coefficients of x^4 and x^5 are p and q respectively.

(b) Find the value of $\dfrac{q}{p}$. (3)

Solution

$$\begin{aligned}(1+x)^{40} &= \ldots + \binom{40}{4}x^4 + \binom{40}{5}x^5 + \ldots \\ &= \ldots + 91\,390x^4 + 658\,008x^5 + \ldots\end{aligned}$$

and

$$\frac{q}{p} = \frac{658\,008}{91\,390} = \underline{\underline{\frac{36}{5}}}.$$

14. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + \frac{1}{4}x)^8$, giving each term in its simplest form. (4)

Solution

$$\begin{aligned}(1 + \tfrac{1}{4}x)^8 &= 1 + \binom{8}{1}(\tfrac{1}{4}x)^1 + \binom{8}{2}(\tfrac{1}{4}x)^2 + \binom{8}{3}(\tfrac{1}{4}x)^3 + \dots \\ &= \underline{\underline{1 + 2x + \tfrac{7}{4}x^2 + \tfrac{7}{8}x^3 + \dots}}\end{aligned}$$

(b) Use your expansion to estimate the value of $(1.025)^8$, giving your answer to 4 decimal places. (3)

Solution

$$\begin{aligned}(1.025)^8 &= (1 + \tfrac{1}{4} \times 0.1)^{10} \\ &\approx 1 + 2(0.1) + \tfrac{7}{4}(0.1)^2 + \tfrac{7}{8}(0.1)^3 \\ &= 1.218\,375 \\ &= \underline{\underline{1.218\,4 \text{ (5 dp)}}}.\end{aligned}$$

15. Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2 - 3x)^5$, giving each term in its simplest form. (4)

Solution

$$\begin{aligned}(2 - 3x)^5 &= [2 + (-3x)]^5 \\ &= 2^5 + \binom{5}{1}2^4(-3x)^1 + \binom{5}{2}2^3(-3x)^2 + \dots \\ &= \underline{\underline{32 - 576x + 540x^2 + \dots}}\end{aligned}$$

16. Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2 - 3x)^5$. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(2-5x)^6 &= [2+(-5x)]^6\\ &= 2^6 + \binom{6}{1}2^5(-5x)^1 + \binom{6}{2}2^4(-5x)^2 + \ldots\\ &= \underline{\underline{64 - 960x + 6000x^2 + \ldots}}\end{aligned}$$

17. (a) Use the binomial expansion to find all the terms of the expansion of $(2+3x)^4$. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(2+3x)^4 &= 2^4 + \binom{4}{1}2^3(3x)^1 + \binom{4}{2}2^2(3x)^2 + \binom{4}{3}(2)(3x)^3 + (3x)^4\\ &= \underline{\underline{16 + 96x + 216x^2 + 216x^3 + 81x^4}}.\end{aligned}$$

(b) Write down the expansion of (1)

$$(2-3x)^4$$

in ascending powers of x, giving each term in its simplest form.

Solution

$$(2-3x)^4 = \underline{\underline{16 - 96x + 216x^2 - 216x^3 + 81x^4}}.$$

18. Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(2-3x)^5$. Give each term in its simplest form. (4)

Solution

$$\begin{aligned}(2-\tfrac{1}{2}x)^8 &= \left[2+(-\tfrac{1}{2}x)\right]^8\\ &= 2^8 + \binom{8}{1}2^7(-\tfrac{1}{2}x)^1 + \binom{8}{2}2^6(-\tfrac{1}{2}x)^2 + \binom{8}{3}2^5(-\tfrac{1}{2}x)^3 + \ldots\\ &= \underline{\underline{256 - 512x + 448x^2 - 224x^3 + \ldots}}\end{aligned}$$

19. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of (4)

$$(2 - 3x)^6,$$

giving each term in its simplest form.

Solution

$$\begin{aligned}(2 - 3x)^6 &= [2 + (-3x)]^6 \\ &= 2^6 + \binom{6}{1} 2^5 (-3x)^1 + \binom{6}{2} 2^4 (-3x)^2 + \dots \\ &= \underline{\underline{64 - 576x + 2160x^2 + \dots}}\end{aligned}$$

(b) Hence, or otherwise, find the first 3 terms, in ascending powers of x, of the binomial expansion of (3)

$$(1 + \tfrac{1}{2})(2 - 3x)^6.$$

Solution

	64	$-576x$	$+2160x^2$
1	64	$-576x$	$+2160x^2$
$+\frac{1}{2}x$	$+32x$	$288x^2$	$\dots$

So

$$(1 + \tfrac{1}{2})(2 - 3x)^6 = \underline{\underline{64 - 544x + 1872x^2 + \dots}}.$$

20. Find the first 4 terms, in ascending powers of x, of the binomial expansion of (4)

$$(1 + \tfrac{3}{2}x)^8,$$

giving each term in its simplest form.

Solution

$$\begin{aligned}(1 + \tfrac{3}{2}x)^8 &= 1 + \binom{8}{1}(\tfrac{3}{2}x)^1 + \binom{8}{2}(\tfrac{3}{2}x)^2 + \binom{8}{3}(\tfrac{3}{2}x)^2 + \dots \\ &= \underline{\underline{1 + 12x + 63x^2 + 189x^3 + \dots}}\end{aligned}$$

21. Find the first 3 terms, in ascending powers of x, of the binomial expansion of (4)

$$(2 - \tfrac{1}{4}x)^{10},$$

giving each term in its simplest form.

Solution

$$\begin{aligned}(2 - \tfrac{1}{4}x)^{10} &= \left[2 + (-\tfrac{1}{4}x)\right]^{10} \\ &= 2^{10} + \binom{10}{1}2^9(-\tfrac{1}{4}x)^1 + \binom{10}{2}2^8(-\tfrac{1}{4}x)^2 + \ldots \\ &= \underline{\underline{1024 - 1280x + 720x^2 + \ldots}}\end{aligned}$$

22. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of (4)

$$(2 - 9x)^{10},$$

giving each term in its simplest form.

Solution

$$\begin{aligned}(2 - 9x)^4 &= [2 + (-9x)]^4 \\ &= 2^4 + \binom{4}{1}2^2(-9x)^1 + \binom{4}{2}2^8(-9x)^2 + \ldots \\ &= \underline{\underline{16 - 288x + 1944x^2 + \ldots}}\end{aligned}$$

$$f(x) = (1 + kx)(2 - 9x)^{10}, \text{ where } k \text{ is a constant.}$$

The expansion, in ascending powers of x, of $f(x)$ up to and including the term in x^2 is

$$A - 232x + Bx^2,$$

where A and B are constants.

(b) Write down the value of A. (1)

Solution

	16	$-288x$	$+1944x^2$
1	16	$-288x$	$+1944x^2$
$+kx$	$+16kx$	$-288kx^2$	$\ldots$

$\underline{\underline{A = 16}}$.

(c) Find the value of k. (2)

Solution

$$16k - 288 = -232 \Rightarrow 16k = 56 \Rightarrow \underline{\underline{k = \tfrac{7}{2}}}.$$

(d) Hence find the value of B. (2)

Solution

$$B = 1944 - 288 \times \tfrac{7}{2} \Rightarrow \underline{\underline{B = 936}}.$$

23. In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the the coefficient of x^3 is equal to the coefficient of x^2.

(a) Prove that $n = 6k + 2$. (6)

Solution

$$\begin{aligned}(2k + x)^n &= \ldots + \binom{n}{2}(2k)^{n-2}x^2 + \binom{n}{3}(2k)^{n-3}x^3 + \ldots \\ &= \ldots + \tfrac{1}{2}n(n-1)(2k)^{n-2}x^2 + \tfrac{1}{6}n(n-1)(n-2)(2k)^{n-3}x^3 + \ldots\end{aligned}$$

and

$$\begin{aligned} & \tfrac{1}{2}n(n-1)(2k)^{n-2} = \tfrac{1}{6}n(n-1)(n-2)(2k)^{n-3} \\ \Rightarrow\ & \tfrac{1}{2}n(n-1)(2k)^{n-2} - \tfrac{1}{6}n(n-1)(n-2)(2k)^{n-3} = 0 \\ \Rightarrow\ & \tfrac{1}{6}n(n-1)(2k)^{n-3}[6k - (n-2)] = 0 \\ \Rightarrow\ & 6k - (n-2) = 0 \\ \Rightarrow\ & \underline{\underline{n = 6k + 2}},\end{aligned}$$

unless $n = 1$ or $n = 2$ (why?).

(b) Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to the term in x^3, giving each term as an exact fraction in its simplest form. (5)

Solution

$$\begin{aligned}(\tfrac{4}{3} + x)^6 &= \left(\tfrac{4}{3}\right)^6 + \binom{6}{1}\left(\tfrac{4}{3}\right)^5 x^1 + \binom{6}{2}\left(\tfrac{4}{3}\right)^4 x^2 + \binom{6}{3}\left(\tfrac{4}{3}\right)^3 x^3 + \ldots \\ &= \underline{\underline{\tfrac{4096}{729} + \tfrac{2048}{81}x + \tfrac{1280}{27}x^2 + \tfrac{1280}{27}x^3 + \ldots}}\end{aligned}$$

24. Find the first 4 terms, in ascending powers of x, of the binomial expansion of (4)

$$\left(3 - \tfrac{1}{3}x\right)^5,$$

giving each term in its simplest form.

Solution

$$\begin{aligned}\left(3 - \tfrac{1}{3}x\right)^5 &= \left[3 + (-\tfrac{1}{3}x)\right]^5 \\ &= 3^5 + \binom{5}{1}3^4(-\tfrac{1}{3}x)^1 + \binom{5}{2}3^3(-\tfrac{1}{3}x)^2 + \binom{5}{2}3^2(-\tfrac{1}{3}x)^3 + \ldots \\ &= \underline{\underline{243 - 135x + 30x^2 - \tfrac{10}{3}x^3 + \ldots}}\end{aligned}$$