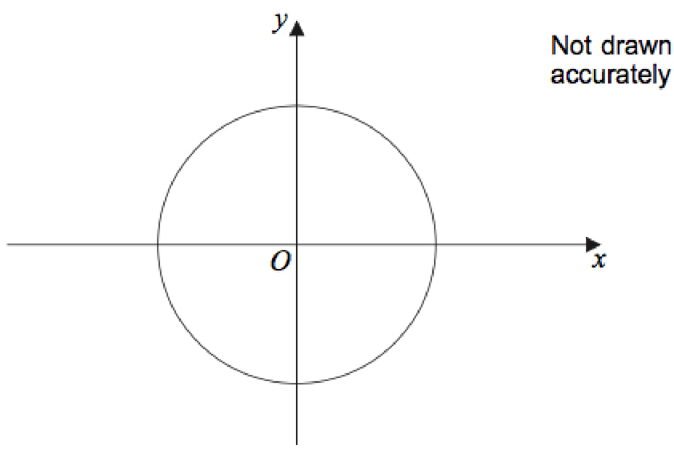
Dr Oliver Mathematics
AQA Further Maths Level 2
June 2012 Paper 2
2 hours

The total number of marks available is 105.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. Here is a sketch of the circle (3)

$$x^2 + y^2 = 36.$$



Work out the circumference of the circle.

Solution

$$x^2 + y^2 = 36 = 6^2$$

and so the radius $r = 6$. Finally,

$$\begin{aligned} \text{circumference} &= 2 \times \pi \times 6 \\ &= \underline{\underline{12\pi \text{ or } 37.7 \text{ cm (3 sf)}}}. \end{aligned}$$

2. (2)

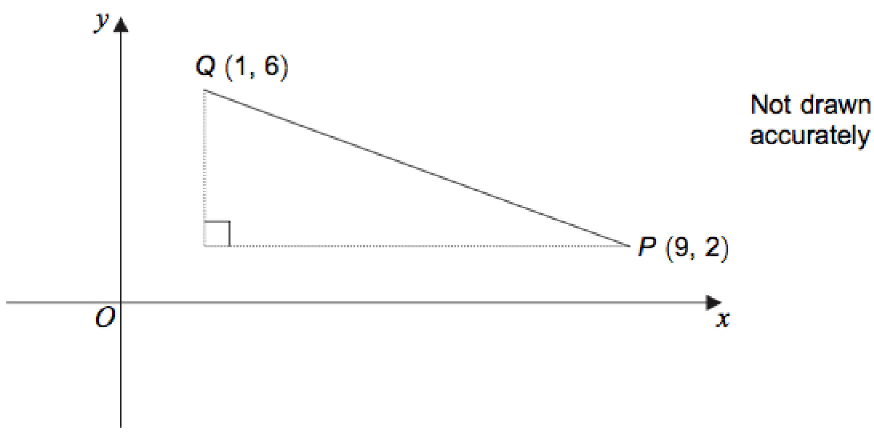
$$y = 5x^3 - 4x^2.$$

Work out $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = 5x^3 - 4x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 - 8x.$$

3. Here is a picture. (4)



Work out the length of PQ.
Give your answer to 3 significant figures.

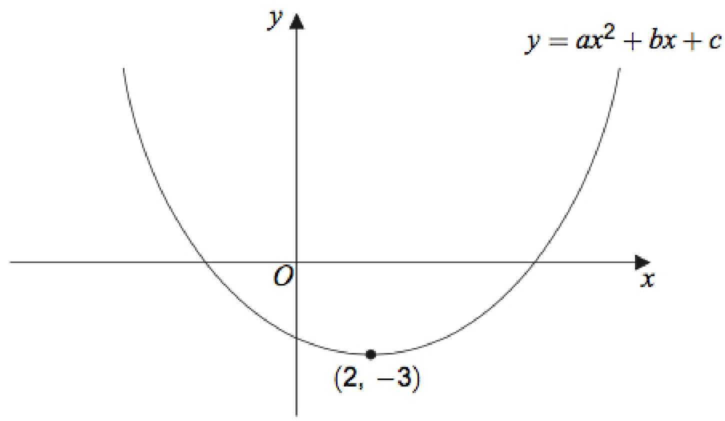
Solution

$$\begin{aligned} PQ &= \sqrt{(9-1)^2 + (2-6)^2} \\ &= \sqrt{8^2 + (-4)^2} \\ &= \sqrt{64 + 16} \\ &= \sqrt{80} \\ &= 8.944\,271\,91 \text{ (FCD)} \\ &= 8.94 \text{ (3 sf)}. \end{aligned}$$

4. A sketch of

$$y = ax^2 + bx + c$$

is shown. The minimum point is $(2, -3)$.



For the sketch shown, circle the correct answer in each of the following.

(a) The value of a is (1)

zero positive negative

Solution
Positive

(b) The value of c is (1)

zero positive negative

Solution
Negative

(c) The solutions of (1)

$$ax^2 + bx + c = 0$$

are

both zero both positive both negative one positive and one negative

Solution
One positive and one negative.

(d) The **number** of solutions of (1)

$$ax^2 + bx + c = -6$$

is

$$0 \qquad 1 \qquad 2 \qquad 3$$

Solution

$\underline{\underline{0}}$.

(e) The equation of the tangent to (1)

$$y = ax^2 + bx + c$$

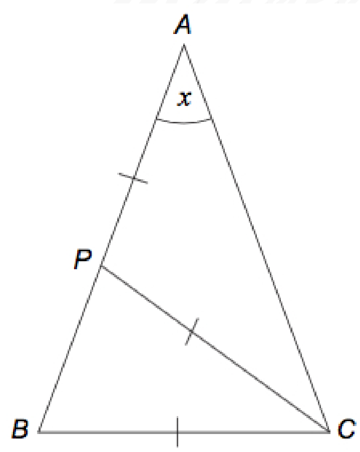
at $(2, -3)$ is

$$x = 2 \qquad y = 2 \qquad x = -3 \qquad y = -3$$

Solution

$\underline{\underline{y = -3}}$.

5. ABC is a triangle.
P is a point on AB such that $AP = PC = BC$.
Angle $BAC = x$.



Not drawn accurately

(a) Prove that angle $ABC = 2x$. (3)

Solution

$\angle PCA = x$ (base angles)
$\angle CPA = 180 - 2x$ (complete the triangle)
$\angle PBC = 2x$ (supplementary angles)
$\angle ABC = \underline{\underline{2x}}$ (base angles)

You are also given that $AB = AC$.

(b) Work out the value of x. (3)

Solution

Well, we note $\angle ACB = 2x$ (base angles in $\triangle ABC$) and

$$x + 2x + 2x = 180 \Rightarrow 5x = 180$$
$$\Rightarrow \underline{\underline{x = 36^\circ}}.$$

6. (a) Expand (2)

$$3x(2x - 5y).$$

Solution

$$3x(2x - 5y) = \underline{\underline{6x^2 - 15xy}}.$$

(b) Expand (3)

$$(3x + 2y)(3x - 4y).$$

Solution

$\times$	$3x$	$+2y$
$3x$	$9x^2$	$+6xy$
$-4y$	$-12xy$	$-8y^2$

$$(3x + 2y)(3x - 4y) = \underline{\underline{9x^2 - 6xy - 8y^2}}.$$

(c) Work out the ratio (2)

$$(3x + 2y)(3x - 4y) : 3x(2x - 5y)$$

when $y = 0$.
Give your answer as simply as possible.

Solution

$$\begin{aligned}(3x + 2y)(3x - 4y) : 3x(2x - 5y) &= (9x^2 - 6xy - 8y^2) : (6x^2 - 15xy) \\ &= 9x^2 : 6x^2 \\ &= \underline{\underline{3 : 2}}.\end{aligned}$$

7.

$$1 \leqslant m \leqslant 5 \text{ and } -9 \leqslant n \leqslant 2.$$

(a) Work out an inequality for $m + n$. (2)

Solution

$$\underline{\underline{-8 \leqslant m + n \leqslant 7}}.$$

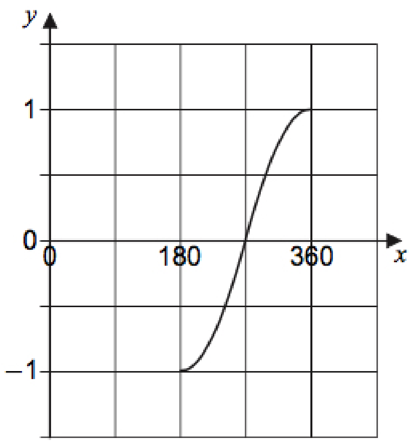
(b) Work out an inequality for $(m + n)^2$. (2)

Solution

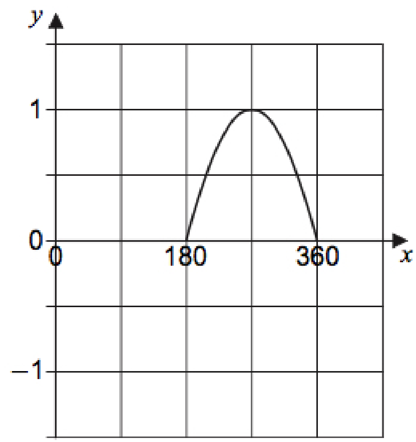
$$\underline{\underline{0 \leqslant (m + n)^2 \leqslant 64}}.$$

8. Four graphs are shown for $180^\circ \leqslant x \leqslant 360^\circ$.

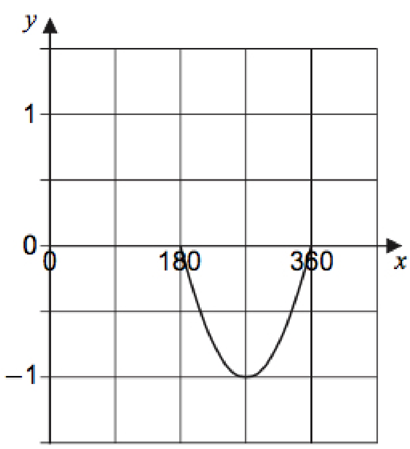
Graph A



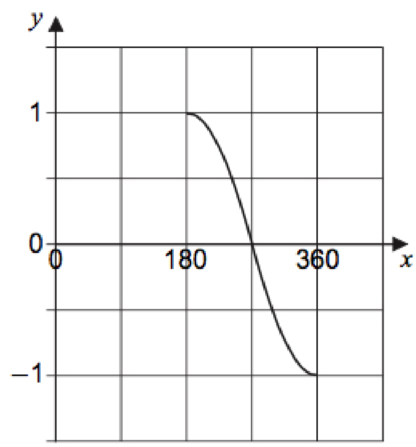
Graph B



Graph C



Graph D



(a) Which graph is $y = \sin x$? (1)

Solution
Graph C.

(b) Which graph is $y = \cos x$? (1)

Solution
Graph A.

9. Here is a formula:

$$5t + 3 = 4w(t + 2).$$

(a) Rearrange the formula to make t the subject. (4)

Solution

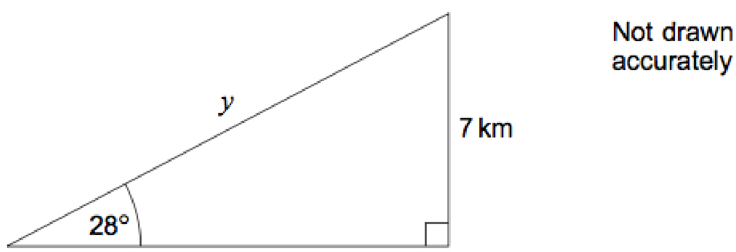
$$\begin{aligned} 5t + 3 = 4w(t+2) &\Rightarrow 5t + 3 = 4tw + 8w \\ &\Rightarrow 5t - 4tw = 8w - 3 \\ &\Rightarrow t(5 - 4w) = 8w - 3 \\ &\Rightarrow \underline{\underline{t = \frac{8w - 3}{5 - 4w}}}. \end{aligned}$$

(b) Work out the exact value of t when $w = -\frac{1}{8}$. (3)
Give your answer in its simplest form.

Solution

$$\begin{aligned} w = -\tfrac{1}{8} \Rightarrow t &= \frac{8(-\frac{1}{8}) - 3}{5 - 4(-\frac{1}{8})} \\ \Rightarrow t &= \frac{-1 - 3}{5 + \frac{1}{2}} \\ \Rightarrow t &= \frac{-4}{5\frac{1}{2}} \\ \Rightarrow \underline{\underline{t}} &\underline{\underline{= -\tfrac{8}{11}}}. \end{aligned}$$

10. An aircraft flies y kilometres in a straight line at an angle of elevation of 28°. (3)
The gain in height is 7 kilometres.



Work out the value of y.

Solution

$$\begin{aligned}\sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin 28^\circ = \frac{7}{y}\\ &\Rightarrow y = \frac{7}{\sin 28^\circ}\\ &\Rightarrow y = 14.910\,381\,28 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{y = 14.9 \text{ km (3 sf)}}}.\end{aligned}$$

11. A sphere has radius x centimetres. (3)
A hemisphere has radius y centimetres.
The shapes have equal volumes.
Work out the value of $\frac{y}{x}$.
Give your answer in the form $a^{\frac{1}{3}}$ where a is an integer.

Solution

Well, the sphere have volume

$$\tfrac{4}{3}\pi x^3$$

and the hemisphere have volume

$$\tfrac{2}{3}\pi y^3.$$

Now,

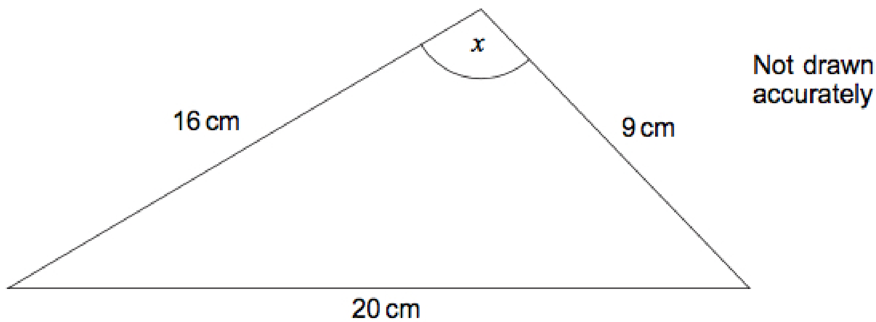
$$\begin{aligned}\tfrac{4}{3}\pi x^3 = \tfrac{2}{3}\pi y^3 &\Rightarrow 2x^3 = y^3\\ &\Rightarrow \frac{y^3}{x^3} = 2\\ &\Rightarrow \left(\frac{y}{x}\right)^3 = 2\\ &\Rightarrow \underline{\underline{\frac{y}{x} = 2^{\frac{1}{3}}}}.\end{aligned}$$

12. Expand and simplify $(t+4)^3$. (3)

Solution

$$(t+4)^3 = t^3 + \binom{3}{1}(t)^2(4) + \binom{3}{2}(t)(4)^2 + 4^3$$
$$= \underline{\underline{t^3 + 12t^2 + 48t + 64}}.$$

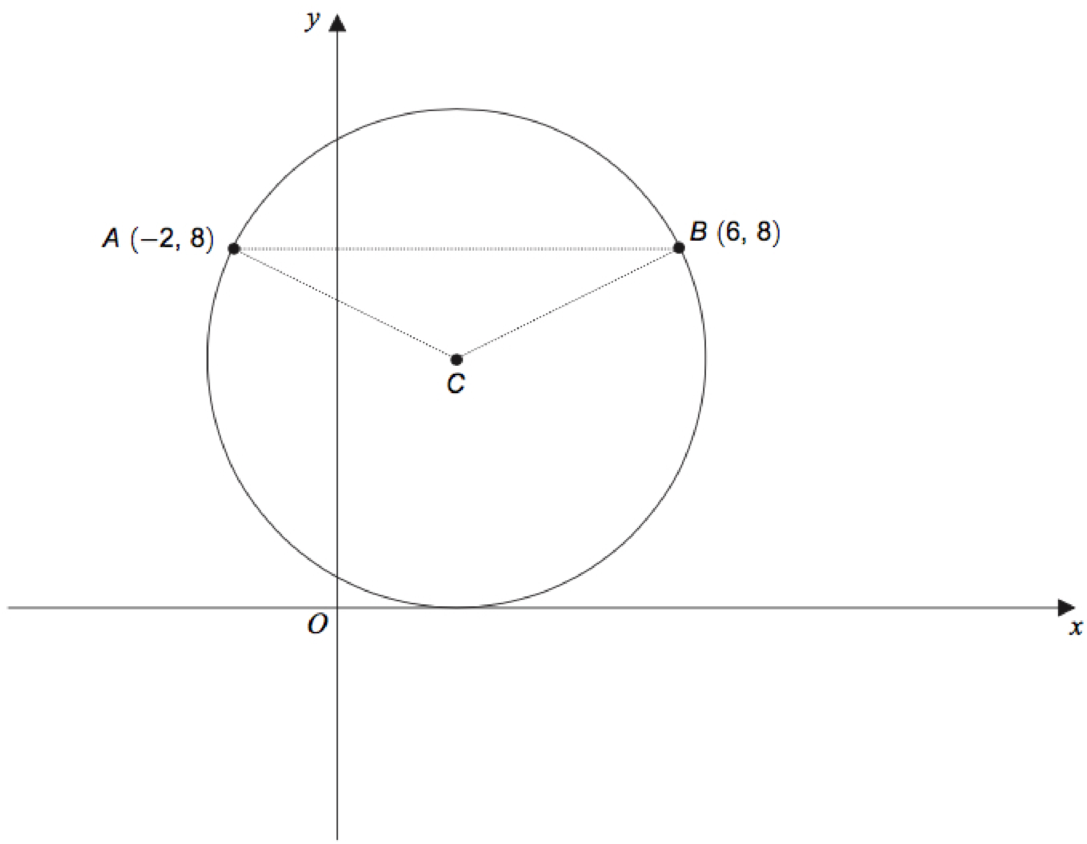
13. Work out angle x. (3)



Solution

$$\cos x = \frac{9^2 + 16^2 - 20^2}{2 \times 9 \times 16} \Rightarrow \cos x = -\tfrac{7}{32}$$
$$\Rightarrow x = 102.635\,625\,1 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 103^\circ \text{ (3 sf)}}}.$$

14. The sketch shows a circle, centre C, radius 5. (4)
The circle passes through the points $A(-2, 8)$ and $B(6, 8)$.
The x-axis is a tangent to the circle.

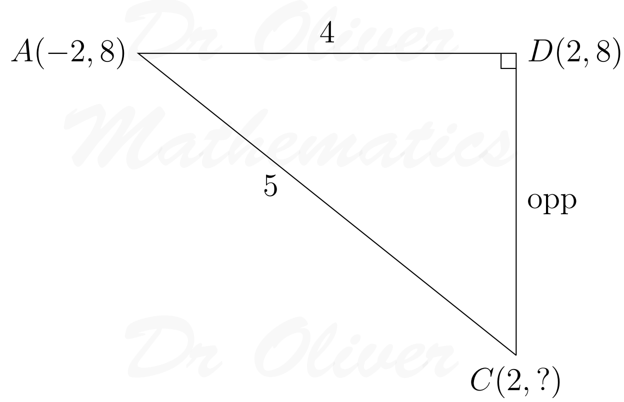


Work out the equation of the circle.

Solution

$$\frac{-2+6}{2} = 2$$

so the point $D(2, 8)$ lies on AB and is directly below the centre.



$$\begin{aligned}\text{hyp}^2 = \text{opp}^2 + \text{adj}^2 &\Rightarrow 5^2 = \text{opp}^2 + 4^2\\ &\Rightarrow \text{opp}^2 = 25 - 16\\ &\Rightarrow \text{opp}^2 = 9\\ &\Rightarrow \text{opp} = 3,\end{aligned}$$

so $C(2, 5)$. Finally, the equation of the circle is

$$\underline{\underline{(x-2)^2 + (y-5)^2 = 25}}.$$

15. (a) (4)

$$\text{f}(x) = 3x - 5 \text{ for all values of } x.$$

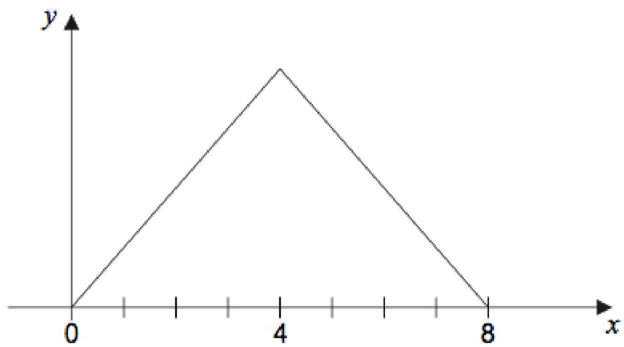
Solve

$$\text{f}(x^2) = 43.$$

Solution

$$\begin{aligned}\text{f}(x^2) = 43 &\Rightarrow 3(x^2) - 5 = 43\\ &\Rightarrow 3x^2 = 48\\ &\Rightarrow x^2 = 16\\ &\Rightarrow \underline{\underline{x = \pm 4}}.\end{aligned}$$

A sketch of $y = \mathrm{g}(x)$ for domain $0 \leqslant x \leqslant 8$ is shown.



The graph is symmetrical about $x = 4$.
The range of $\mathrm{g}(x)$ is $0 \leqslant \mathrm{g}(x) \leqslant 12$.

(b) Work out the function $\mathrm{g}(x)$. (5)

Solution

The gradient of the line joining the origin and the highest point on the graph is

$$\frac{12}{4} = 3$$

and so

$$\mathrm{g}(x) = \begin{cases} 3x, & 0 \leqslant x \leqslant 4 \\ -3x + c, & 4 < x \leqslant 8. \end{cases}$$

Now, it goes through $(8, 0)$:

$$-3(8) + c = 0 \Rightarrow c = 24$$

and we conclude that

$$\mathrm{g}(x) = \begin{cases} \underline{\underline{3x}}, & 0 \leqslant x \leqslant 4 \\ \underline{\underline{-3x + 24}}, & 4 < x \leqslant 8. \end{cases}$$

16. (a) Use the factor theorem to show that $(x - 1)$ and $(x - 4)$ are factors of (2)

$$x^3 - 21x + 20.$$

Solution

Let
$$f(x) \equiv x^3 - 21x + 20.$$
Then,
$$\begin{aligned} f(1) &= 1 - 21 - 20 = 0 \\ f(4) &= 64 - 84 - 20 = 0; \end{aligned}$$
hence, $(x-1)$ and $(x-4)$ are factors.

(b) Show that $(x-1)$ and $(x-4)$ are also factors of (2)
$$x^3 - 10x^2 + 29x - 20.$$

Solution

Let
$$g(x) \equiv x^3 - 10x^2 + 29x - 20.$$
Then,
$$\begin{aligned} g(1) &= 1 - 10 + 29 - 21 - 20 = 0 \\ g(4) &= 64 - 160 + 116 - 20 = 0; \end{aligned}$$
hence, $(x-1)$ and $(x-4)$ are factors.

(c) Hence, simplify fully (3)
$$\frac{x^3 - 21x + 20}{x^3 - 10x^2 + 29x - 20}.$$

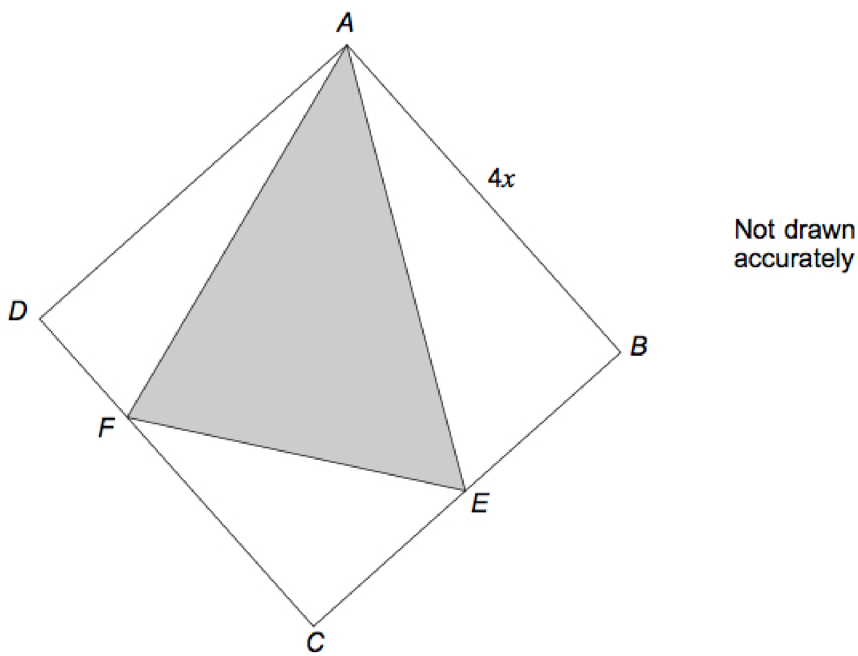
Solution

Well,
$$\frac{20}{1 \times 4} = 5$$
so the remaining factor of $f(x)$ is $(x+5)$ and the remaining factor of $g(x)$ is

$(x-5)$. Hence,

$$\frac{x^3 - 21x + 20}{x^3 - 10x^2 + 29x - 20} = \frac{(x-1)(x-4)(x+5)}{(x-1)(x-4)(x-5)} = \underline{\underline{\frac{x+5}{x-5}}}.$$

17. $ABCD$ is a square of side length $4x$. (5)
E is the midpoint of BC.
$DF : FC = 1 : 3$.



You are given that

$$\text{area of triangle } AEF = kx^2.$$

Work out the value of k.

Solution

$$\begin{aligned}\text{area of triangle } AEF &= \text{area of square} - 3 \times \text{area of little triangles} \\ &= (4x)^2 - (\tfrac{1}{2} \times 2x \times 3x) - (\tfrac{1}{2} \times x \times 4x) - (\tfrac{1}{2} \times 2x \times 4x) \\ &= 16x^2 - 3x^2 - 2x^2 - 4x^2 \\ &= \underline{\underline{7x^2}};\end{aligned}$$

hence, $\underline{\underline{k = 7}}$.

18. (3)

$$(x-5)^2 + a \equiv x^2 + bx + 28.$$

Work out the values of a and b.

Solution

$\times$	x	-5
x	x^2	$-5x$
-5	$-5x$	$+25$

$$\begin{aligned}(x-5)^2 + a \equiv x^2 + bx + 28 &\Rightarrow (x^2 - 10x + 25) + a \equiv x^2 + bx + 28 \\ &\Rightarrow x^2 - 10x + (25 + a) \equiv x^2 + bx + 28.\end{aligned}$$

Hence,

$$\underline{\underline{b = -10}} \text{ and } 25 + a = 3 \Rightarrow \underline{\underline{a = 3}}.$$

19. Solve the simultaneous equations: (6)

$$\begin{aligned}x + y &= 4 \\ y^2 &= 4x + 5.\end{aligned}$$

Do **not** use trial and improvement.

Solution

$$x + y = 4 \Rightarrow y = 4 - x$$

and

$\times$	4	$-x$
4	16	$-4x$
$-x$	$-4x$	$+x^2$

Now,

$$\begin{aligned} y^2 = 4x + 5 &\Rightarrow (4 - x)^2 = 4x + 5 \\ &\Rightarrow 16 - 8x + x^2 = 4x + 5 \\ &\Rightarrow x^2 - 12x + 11 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -12 \\ \text{multiply to:} & +11 \end{array}\right\} -11, -1$$

$$\begin{aligned} &\Rightarrow (x - 1)(x - 11) = 0 \\ &\Rightarrow x = 1 \text{ or } x = 11 \\ &\Rightarrow y = 3 \text{ or } y = -7; \end{aligned}$$

hence,

$$\underline{\underline{x = 1,\ y = 3}} \text{ or } \underline{\underline{x = 11,\ y = -7}}.$$

20. For what values of x is (4)

$$y = 150x - 2x^3$$

an increasing function?

Solution

$$y = 150x - 2x^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 150 - 6x^2$$

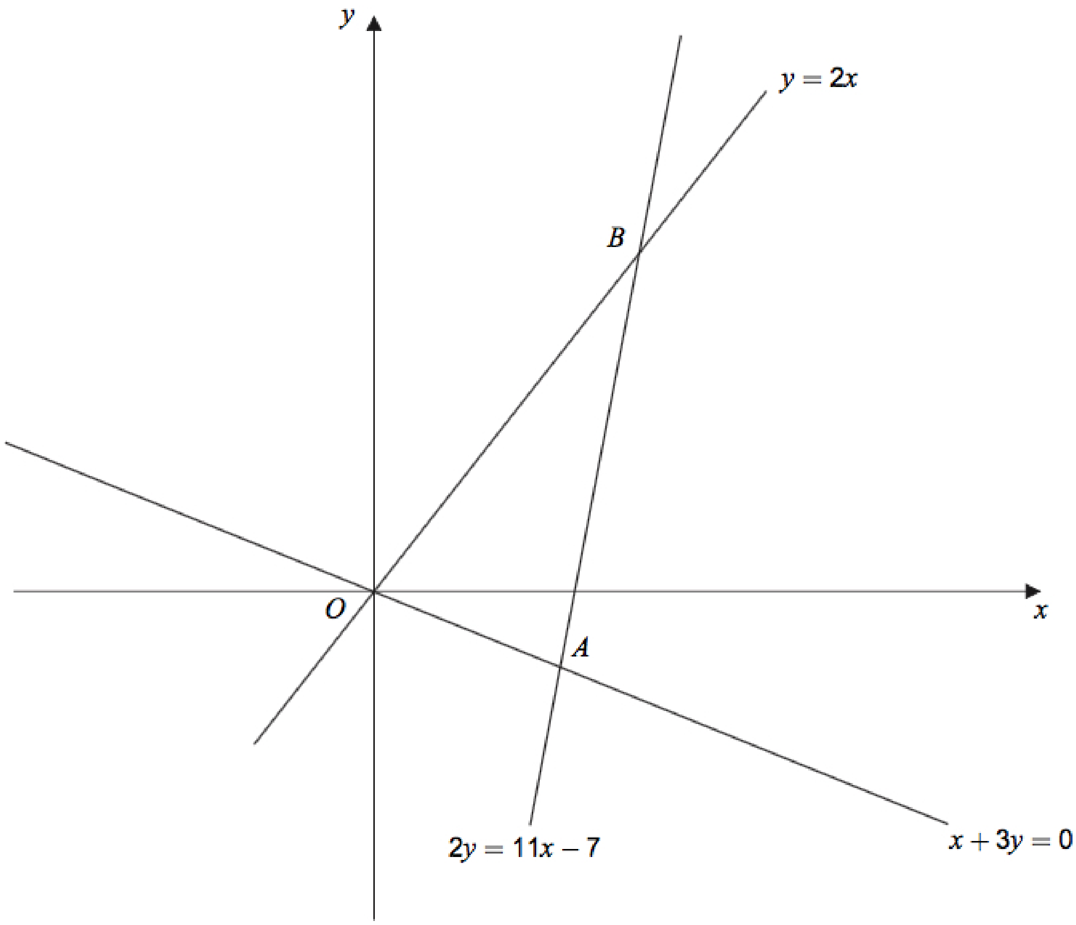
and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} > 0 &\Rightarrow 150 - 6x^2 > 0 \\ &\Rightarrow 150 > 6x^2 \\ &\Rightarrow 25 > x^2 \\ &\Rightarrow \underline{\underline{-5 < x < 5}}. \end{aligned}$$

21. The equations of three straight lines are (6)

$$y = 2x \qquad x + 3y = 0 \qquad 2y = 11x - 7.$$

The lines intersect at the points O, A, and B as shown on this sketch.



Show that

$$\text{length } OB = \text{length } AB.$$

Solution

First,

$$2y = 11x - 7 \Rightarrow y = \tfrac{11}{2}x - \tfrac{7}{2}$$

and

$$x + 3y = 0 \Rightarrow y = -\tfrac{1}{3}x.$$

Second, we work out the point B:

$$2x = \tfrac{11}{2}x - \tfrac{7}{2} \Rightarrow \tfrac{7}{2}x = \tfrac{7}{2}$$
$$\Rightarrow x = 1$$
$$\Rightarrow y = 2$$

and

$$OB = \sqrt{1^2 + 2^2}$$
$$= \sqrt{5}.$$

Third, we work out the point A:

$$\tfrac{11}{2}x - \tfrac{7}{2} = -\tfrac{1}{3}x \Rightarrow \tfrac{35}{6}x = \tfrac{7}{2}$$
$$\Rightarrow x = 0.6$$
$$\Rightarrow y = -0.2$$

and

$$AB = \sqrt{(1 - 0.6)^2 + [2 - (-0.2)^2}$$
$$= \sqrt{5}.$$

Hence,

$$\underline{\underline{\text{length } OB = \text{length } AB}}.$$

22. The transformation matrix (5)

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

maps point P to point Q.
The transformation matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

maps point Q to point R.
Point R is $(-4, 3)$.
Work out the coordinates of point P.

Solution

$$
\begin{aligned}
& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ -3 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.
\end{aligned}
$$

Hence, the answer is

$$\underline{\underline{P(3, 4)}}.$$

23. The curve $y = \mathrm{f}(x)$ is such that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x(x-2)^2.$$

The stationary points of the curve are at $(0, \frac{4}{3})$ and $(2, 0)$.
Determine the nature of each stationary point.
You **must** show your working.

Solution

$\times$	x	-2
x	x^2	$-2x$
-2	$-2x$	$+4$

Now,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = -x(x-2)^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -x(x^2 - 4x + 4) \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -x^3 + 4x^2 - 4x \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -3x^2 + 8x - 4.
\end{aligned}
$$

Now,

$$x = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4 < 0$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Finally, $(0, \frac{4}{3})$ is a maximum turning point and $(2, 0)$ is a point of inflexion.