Dr Oliver Mathematics
Advance Level Further Mathematics
Further Pure Mathematics 2: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. A complex number $z = x + \mathrm{i}y$ is represented by the point P in an Argand diagram.

   Given that
   $$|z - 3| = 4|z + 1|,$$

   (a) show that the locus of P has equation (2)
   $$15x^2 + 15y^2 + 38x + 7 = 0.$$

   Solution

   $$\begin{aligned} |z - 3| = 4|z + 1| &\Rightarrow |(x + \mathrm{i}y) - 3| = 4|(x + \mathrm{i}y) + 1| \\ &\Rightarrow |(x - 3) + \mathrm{i}y|^2 = 16|(x + 1) + \mathrm{i}y)|^2 \\ &\Rightarrow (x - 3)^2 + y^2 = 16[(x + 1) + y^2] \\ &\Rightarrow (x^2 - 6x + 9) + y^2 = 16(x^2 + 2x + 1) + 16y^2 \\ &\Rightarrow x^2 - 6x + 9 + y^2 = 16x^2 + 32x + 16 + 16y^2 \\ &\Rightarrow \underline{\underline{15x^2 + 15y^2 + 38x + 7 = 0}}, \end{aligned}$$

   as required.

   (b) Hence find the maximum value of $|z|$. (3)

   Solution

   $$\begin{aligned} 15x^2 + 15y^2 + 38x + 7 = 0 &\Rightarrow x^2 + \tfrac{38}{15}x + y^2 = -\tfrac{7}{15} \\ &\Rightarrow x^2 + \tfrac{38}{15}x + \left(\tfrac{19}{15}\right)^2 + y^2 = -\tfrac{7}{15} + \left(\tfrac{19}{15}\right)^2 \\ &\Rightarrow (x + \tfrac{19}{15})^2 + y^2 = \tfrac{256}{225} \\ &\Rightarrow (x + \tfrac{19}{15})^2 + y^2 = (\tfrac{16}{15})^2; \end{aligned}$$

hence, the centre is $(-\frac{19}{15}, 0)$ and the radius is $\frac{16}{15}$. Finally,

$$\begin{aligned}\max |z| &= \tfrac{19}{15} + \tfrac{16}{15} \\ &= \underline{\underline{2\tfrac{1}{3}}}.\end{aligned}$$

2. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

(a) Show that 2 is a repeated eigenvalue of $\mathbf{A}$ and find the other eigenvalue. (5)

Solution

$$\begin{aligned}
& \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\
\Rightarrow\quad & \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \\
\Rightarrow\quad & (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0 \\
\Rightarrow\quad & (6-\lambda)[(9 - 6\lambda + \lambda^2) - 1] + 2[-6 + 2\lambda + 2] + 2[2 - 6 + 2\lambda] = 0 \\
\Rightarrow\quad & (6-\lambda)(8 - 6\lambda + \lambda^2) - 8 + 4\lambda - 8 + 4\lambda = 0
\end{aligned}$$

$\times$	8	-6λ	$+\lambda^2$
6	48	-36λ	$+6\lambda^2$
$-\lambda$	-8λ	$+6\lambda^2$	$-\lambda^3$

$$\begin{aligned}
\Rightarrow\quad & (48 - 44\lambda + 12\lambda^2 - \lambda^3) + 8\lambda - 16 = 0 \\
\Rightarrow\quad & 32 - 36\lambda + 12\lambda^2 - \lambda^3 = 0 \\
\Rightarrow\quad & \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0
\end{aligned}$$

We do synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & \downarrow & 2 & -20 & -32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\begin{aligned} &\Rightarrow \quad (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0 \\ &\Rightarrow \quad (\lambda - 2)^2(\lambda - 8) = 0; \end{aligned}$$

hence, the eigenvalues are $\underline{\underline{2,\ 2,\ \text{and}\ 8}}$.

(b) Hence find three non-parallel eigenvectors of $\mathbf{A}$. (4)

Solution

$\underline{\lambda = 8}$:

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and this means

$$\begin{aligned} -2x - 2y + 2z &= 0 \quad (1) \\ -2x - 5y - z &= 0 \quad (2) \\ 2x - y - 5z &= 0 \quad (3). \end{aligned}$$

Do (1) + (3):

$$\begin{aligned} -3y - 3z = 0 &\Rightarrow y = -z \\ &\Rightarrow 2x - y + 5y = 0 \\ &\Rightarrow 2x = -4y \\ &\Rightarrow x = -2y \end{aligned}$$

and, e.g.,

$$\underline{\underline{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}}$$

is an eigenvector.

$\underline{\lambda = 2}$:

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$2x - y - z = 0 \quad (4)$$

for all three equations. Now, set $z = 0$ and we get

$$2x - y = 0 \Rightarrow 2x = y$$

and we set $x = 0$ and we get

$$-y + z = 0 \Rightarrow y = z.$$

Our eigenvectors are then, e.g.,

$$\underline{\underline{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}}$$

are eigenvectors.

(c) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. (2)

Solution

E.g.,

$$\mathbf{P} = \underline{\underline{\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}}}.$$

3. The number of visits to a website, in any particular month, is modelled as the number of visits received in the previous month plus k times the number of visits received in the month before that, where k is a positive constant.

Given that V_n is the number of visits to the website in month n,

(a) write down a general recurrence relation for V_{n+2} in terms of V_{n+1}, V_n, and k. (1)

Solution

$$\underline{\underline{V_{n+2} = V_{n+1} + kV_n}}.$$

For a particular website you are given that

- $k = 0.24$,
- In month 1, there were 65 visits to the website, and
- In month 2, there were 71 visits to the website.

(b) Show that (5)

$$V_n = 50(1.2)^n - 25(-0.2)^n.$$

Solution

$$V_{n+2} = V_{n+1} + kV_n \Rightarrow V_{n+2} - V_{n+1} - kV_n = 0.$$

Now, if

$$V_{n+2} = Ar^{n+2}$$

is a solution to the recurrence equation

$$V_{n+2} = aV_{n+1} + bV_n,$$

then

$$\begin{aligned} Ar^{n+2} = Aar^{n+1} + Abr^n &\Rightarrow Ar^{n+2} - Aar^{n+1} - Abr^n = 0 \\ &\Rightarrow Ar^n(r^2 - ar - b) = 0 \\ &\Rightarrow r^2 - ar - b = 0. \end{aligned}$$

Now, here, $a = 1$ and $b = 0.24$:

$$\begin{aligned} r^2 - r - 0.24 = 0 &\Rightarrow (r - 1.2)(r + 0.2) = 0 \\ &\Rightarrow r = 1.2 \text{ or } r = -0.2 \end{aligned}$$

and

$$V_n = a(1.2)^n + b(-0.2)^n.$$

Next,

$$65 = 1.2a - 0.2b \quad (1)$$

and

$$71 = 1.44a + 0.04b \quad (2).$$

Do $1.2 \times (1) - (2)$:

$$\begin{aligned} 7 = -0.28b &\Rightarrow b = -25 \\ &\Rightarrow a = 50 \end{aligned}$$

and, finally, we have

$$\underline{\underline{V_n = 50(1.2)^n - 25(-0.2)^n}}.$$

This model predicts that the number of visits to this website will exceed one million for the first time in month N.

(c) Find the value of N. (2)

Solution

Now, $(-0.2)^n$ is negligible for large n and so

$$\begin{aligned} 50(1.2)^N > 1\,000\,000 &\Rightarrow (1.2)^N > 20\,000 \\ &\Rightarrow \ln(1.2)^N > \ln 20\,000 \\ &\Rightarrow N \ln 1.2 > \ln 20\,000 \\ &\Rightarrow N > \frac{\ln 20\,000}{\ln 1.2} \\ &\Rightarrow N = 54.318\,796\,56 \text{ (FCD)}; \end{aligned}$$

hence, this model predicts that the number of visits to this website will exceed one million for the first time in month $\underline{\underline{55}}$.

4. (a) Use Fermat's Little Theorem to find the least positive residue of 6^{542} modulo 13. (5)

Solution

Since 13 is not divisible by 6,

$$6^{12} \equiv 1 (\text{mod } 13).$$

Now,

$$\begin{aligned} 6^{542} &\equiv 6^{2+45\times 12} \\ &\equiv 6^2\left(6^{12}\right)^{45} \\ &\equiv 36 \\ &\equiv \underline{\underline{10 (\text{mod } 13)}}. \end{aligned}$$

Seven students, Alan, Brenda, Charles, Devindra, Enid, Felix, and Graham, are attending a concert and will sit in a particular row of 7 seats. Find the number of ways they can be seated if

(b) (i) there are no restrictions where they sit in the row, (1)

Solution

$$7! = \underline{\underline{5\,040}}.$$

(ii) Alan, Enid, Felix, and Graham sit together, (2)

Solution
Alan, Enid, Felix, and Graham sit together which makes

$$4! = 24$$

combinations and Brenda, Charles, Devindra, AEFG sit together which makes

$$4! = 24.$$

In total, there are

$$24 \times 24 = \underline{\underline{576}}.$$

(iii) Brenda sits at one end of the row and Graham sits at the other end of the row, (2)

Solution

$$2!\,5! = 2 \times 120 = \underline{\underline{240}}.$$

(iv) Charles and Devindra do not sit together. (2)

Solution

$$\begin{aligned} 7! - 6!\,2! &= 5\,040 - (720 \times 2) \\ &= 5\,040 - 1\,440 \\ &= \underline{\underline{3\,600}}. \end{aligned}$$

5.

$$I_n = \int \operatorname{cosec}^n x \,\mathrm{d}x,\ n \in \mathbb{Z}.$$

(a) Prove that, for $n \geqslant 2$, (4)

$$I_n = \left(\frac{n-2}{n-1}\right) I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}.$$

Solution

$$u = \operatorname{cosec}^{n-2} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -(n-2)\operatorname{cosec}^{n-2} x \cot x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \operatorname{cosec}^2 x \Rightarrow v = -\cot x.$$

Now,

$$\begin{aligned} I_n &= -\operatorname{cosec}^{n-2} x \cot x - \int (n-2)\operatorname{cosec}^{n-2} x \cot^2 x \,\mathrm{d}x \\ &= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x - 1)\,\mathrm{d}x \\ &= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int (\operatorname{cosec}^{n} x - \operatorname{cosec}^{n-2} x)\,\mathrm{d}x \\ &= -\operatorname{cosec}^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2} \end{aligned}$$

and

$$\begin{aligned} & I_n + (n-2)I_n = (n-2)I_{n-2} - \operatorname{cosec}^{n-2} x \cot x \\ \Rightarrow\ & I_n[1 + (n-2)] = (n-2)I_{n-2} - -\operatorname{cosec}^{n-2} x \cot x \\ \Rightarrow\ & (n-1)I_n = (n-2)I_{n-2} - \operatorname{cosec}^{n-2} x \cot x \\ \Rightarrow\ & \underline{\underline{I_n = \left(\frac{n-2}{n-1}\right) I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}}}, \end{aligned}$$

as required.

(b) Hence show that (4)

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \,\mathrm{d}x = \tfrac{56}{135}\sqrt{3}.$$

Solution

Let

$$J_n = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^6 x \,\mathrm{d}x.$$

Then

$$
\begin{aligned}
J_6 &= \tfrac{4}{5}J_4 - \tfrac{1}{5}\left[\operatorname{cosec}^4 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \tfrac{4}{5}\left(\tfrac{2}{3}J_2 - \tfrac{1}{3}\left[\operatorname{cosec}^2 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}}\right) - \tfrac{1}{5}\left[\operatorname{cosec}^4 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \tfrac{8}{15}J_2 - \tfrac{4}{15}\left[\operatorname{cosec}^2 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} - \tfrac{1}{5}\left[\operatorname{cosec}^4 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \tfrac{8}{15}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x \,\mathrm{d}x - \tfrac{4}{15}\left[\operatorname{cosec}^2 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} - \tfrac{1}{5}\left[\operatorname{cosec}^4 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \tfrac{8}{15}\left[-\cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} - \tfrac{4}{15}\left[\operatorname{cosec}^2 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} - \tfrac{1}{5}\left[\operatorname{cosec}^4 x \cot x\right]_{x=\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \tfrac{8}{15}\left(0 + \tfrac{\sqrt{3}}{3}\right) - \tfrac{4}{15}\left(0 - \tfrac{4\sqrt{3}}{9}\right) - \tfrac{1}{5}\left(0 - \tfrac{16\sqrt{3}}{27}\right) \\
&= \tfrac{8\sqrt{3}}{45} + \tfrac{16\sqrt{3}}{135} + \tfrac{16\sqrt{3}}{135} \\
&= \underline{\underline{\tfrac{56}{135}\sqrt{3}}},
\end{aligned}
$$

as required.

6. (a) A binary operation $*$ is defined on positive real numbers by (4)

$$a * b = a + b + ab.$$

Prove that the operation $*$ is associative.

Solution

$$
\begin{aligned}
a * (b * c) &= a * (b + c + bc) \\
&= a + (b + c + bc) + a(b + c + bc) \\
&= a + b + c + bc + ab + ac + abc
\end{aligned}
$$

and

$$
\begin{aligned}
(a * b) * c &= (a + b + ab) * c \\
&= (a + b + ab) + c + c(a + b + ab) \\
&= a + b + ab + c + ac + bc + abc \\
&= a + b + c + bc + ab + ac + abc \\
&= a * (b * c)
\end{aligned}
$$

and, hence, we have proved that the operation $\underline{\underline{* \text{ is associative}}}$.

(b) The set

$$G = \{1, 2, 3, 4, 5, 6\}$$

forms a group under the operation of multiplication modulo 7.

(i) Show that G is cyclic. (2)

Solution

$$\begin{aligned} 3^1 &= 3 \\ 3^2 &= 9 \equiv 2 \\ 3^3 &= 27 \equiv 6 \\ 3^4 &= 81 \equiv 4 \\ 3^5 &= 243 \equiv 5 \\ 3^6 &= 729 \equiv 1 \end{aligned}$$

and so the G is cyclic.

The set

$$H = \{1, 5, 7, 11, 13, 17\}$$

forms a group under the operation of multiplication modulo 18.

(ii) List all the subgroups of H. (3)

Solution

The orders of all the subgroups of H is 1, 2, 3, and 6.

$1^2 = 1$.
$5^2 = 25 \equiv 7$, $5^3 = 125 \equiv 17$, $5^4 = 625 \equiv 13$, $5^5 = 3\,125 \equiv 11$,
$5^6 = 15\,625 \equiv 1$
$7^2 = 49 \equiv 13$, $7^3 = 343 \equiv 1$.
$11^2 = 121 \equiv 13$, $11^3 = 1\,331 \equiv 17$, $11^4 = 14\,641 \equiv 7$, $11^5 = 161\,051 \equiv 5$,
$11^6 = 1\,771\,561 \equiv 1$.
$13^2 = 169 \equiv 13$, $13^3 = 2\,197 \equiv 1$.
$17^2 = 289 \equiv 1$.

Order 1: $\{1\}$.

Order 2: $\{1, 17\}$.

Order 3: $\{1, 7, 13\}$.

Order 6: $\{1, 5, 7, 11, 13, 17\}$.

(iii) Describe an isomorphism between G and H. (3)

Solution
E.g.,

G	1	2	3	4	5	6
H	1	7	5	13	11	17

7. A transformation from the z-plane to the w-plane is given by

$$w = \frac{3\mathrm{i}z - 2}{z + \mathrm{i}},\ z \neq -\mathrm{i}.$$

(a) Show that the circle C with equation (4)

$$|z + \mathrm{i}| = 1$$

in the z-plane is mapped to a circle D in the w-plane, giving a Cartesian equation for D.

Solution

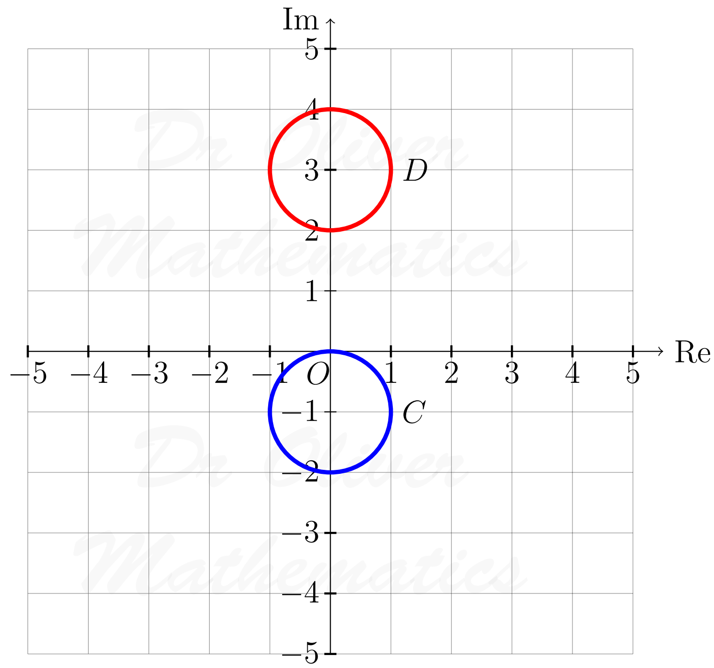
$$\begin{aligned} w = \frac{3\mathrm{i}z - 2}{z + \mathrm{i}} &\Rightarrow w(z + \mathrm{i}) = 3\mathrm{i}z - 2 \\ &\Rightarrow wz + \mathrm{i}w = 3\mathrm{i}z - 2 \\ &\Rightarrow 2 + \mathrm{i}w = 3\mathrm{i}z - wz \\ &\Rightarrow 2 + \mathrm{i}w = z(3\mathrm{i} - w) \\ &\Rightarrow z = \frac{2 + \mathrm{i}w}{3\mathrm{i} - w} \end{aligned}$$

and

$$\begin{aligned} |z + \mathrm{i}| = 1 &\Rightarrow \left|\frac{2 + \mathrm{i}w}{3\mathrm{i} - w} + \mathrm{i}\right| = 1 \\ &\Rightarrow \left|\frac{2 + \mathrm{i}(u + \mathrm{i}v) + \mathrm{i}[3\mathrm{i} - (u + \mathrm{i}v)]}{3\mathrm{i} - (u + \mathrm{i}v)}\right| = 1 \\ &\Rightarrow \left|\frac{2 + \mathrm{i}u - v - 3 - \mathrm{i}u + v}{3\mathrm{i} - u - \mathrm{i}v}\right| = 1 \\ &\Rightarrow \left|\frac{-1}{-u - \mathrm{i}(3 - v)}\right| = 1 \\ &\Rightarrow u^2 + (3 - v)^2 = 1 \\ &\Rightarrow \underline{\underline{u^2 + (v - 3)^2 = 1}}. \end{aligned}$$

(b) Sketch C and D on Argand diagrams. (2)

Solution



8. Figure 1 shows the vertical cross section of a child's spinning top. The point A is vertically above the point B and the height of the spinning top is 5 cm.

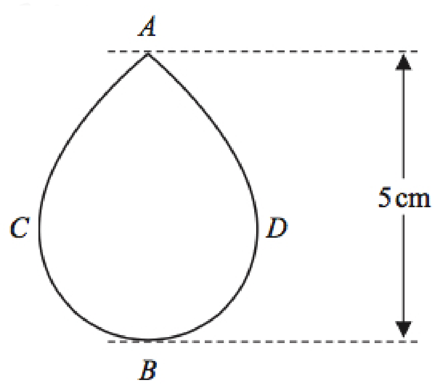


Figure 1: child's spinning top

The line CD is perpendicular to AB such that CD is the maximum width of the spinning top.

The spinning top is modelled as the solid of revolution created when part of the curve with polar equation

$$r^2 = 25\cos 2\theta,$$

is rotated through 2π radians about the initial line.

(a) Show that, according to the model, the surface area of the spinning top is (7)

$$k\pi(2-\sqrt{2})\ \text{cm}^2,$$

where k is a constant to be determined.

Solution

$$\begin{aligned} r^2 = 25\cos 2\theta \Rightarrow r &= 5(\cos 2\theta)^{\frac{1}{2}} \\ \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} &= 5 \cdot \tfrac{1}{2}(\cos 2\theta)^{-\frac{1}{2}} \cdot (-2\sin 2\theta) \\ \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} &= -5(\cos 2\theta)^{-\frac{1}{2}}\sin 2\theta \end{aligned}$$

and

$$\begin{aligned} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} &= \sqrt{25\cos 2\theta + 25(\cos 2\theta)^{-1}\sin^2 2\theta} \\ &= 5\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} \\ &= 5\sqrt{\sec 2\theta}. \end{aligned}$$

Next,

$$2\theta = \tfrac{1}{2}\pi \Rightarrow \theta = \tfrac{1}{4}\pi.$$

Finally,

$$\begin{aligned} \text{surface area} &= 2\pi\int_0^{\frac{1}{4}\pi} r\sin\theta\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= 2\pi\int_0^{\frac{1}{4}\pi} 5(\cos 2\theta)^{\frac{1}{2}}\sin\theta \cdot 5\sqrt{\sec 2\theta}\,\mathrm{d}\theta \\ &= 50\pi\int_0^{\frac{1}{4}\pi}\sin\theta\,\mathrm{d}\theta \\ &= 50\pi\left[-\cos\theta\right]_{\theta=0}^{\frac{1}{4}\pi} \\ &= 50\pi\left(-\tfrac{\sqrt{2}}{2} + 1\right) \\ &= \underline{\underline{25\pi\left(2-\sqrt{2}\right)}}; \end{aligned}$$

hence, $\underline{\underline{k = 25}}$.

(b) Show that, according to the model, the length CD is $\frac{5}{2}\sqrt{2}$. (6)

Solution

$$\begin{aligned}
y = r\sin\theta \Rightarrow y &= 5(\cos 2\theta)^{\frac{1}{2}} \cdot \sin\theta \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \tfrac{5}{2}(\cos 2\theta)^{-\frac{1}{2}}(-2\sin 2\theta) \cdot \sin\theta + 5(\cos 2\theta)^{\frac{1}{2}} \cdot \cos\theta \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{-5\sin 2\theta \sin\theta}{\sqrt{\cos 2\theta}} + 5\sqrt{\cos 2\theta}\cos\theta \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{5(\cos 2\theta\cos\theta - \sin 2\theta\sin\theta)}{\sqrt{\cos 2\theta}} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{5\cos 3\theta}{\sqrt{\cos 2\theta}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 &\Rightarrow \frac{5\cos 3\theta}{\sqrt{\cos 2\theta}} = 0 \\
&\Rightarrow \cos 3\theta = 0 \\
&\Rightarrow 3\theta = \tfrac{1}{2}\pi \\
&\Rightarrow \theta = \tfrac{1}{6}\pi.
\end{aligned}$$

Finally,

$$\begin{aligned}
CD &= 2r\sin\tfrac{1}{6}\pi \\
&= 2 \cdot 5\sqrt{\cos\tfrac{1}{3}\pi} \cdot \tfrac{1}{2} \\
&= \underline{\underline{\tfrac{5}{2}\sqrt{2}\text{ cm}}}.
\end{aligned}$$