Dr Oliver Mathematics
Mathematics: Advanced Higher
2016 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Differentiate (2)
$$y = x \tan^{-1} 2x.$$

(b) Given (3)
$$\mathrm{f}(x) = \frac{1 - x^2}{1 + 4x^2},$$

find $\mathrm{f}'(x)$, simplifying your answer.

A curve is given by the parametric equations

$$x = 6t \text{ and } y = 1 - \cos t.$$

(c) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. (3)

2. A geometric sequence has second and fifth terms 108 and 4 respectively.

(a) Calculate the value of the common ratio. (3)

(b) State why the associated geometric series has a sum to infinity. (1)

(c) Find the value of this sum to infinity. (2)

3. (a) Write down and simplify the general term in the binomial expansion of (1)

$$\left(\frac{3}{x} - 2x \right)^{13}.$$

(b) Hence, or otherwise, find the term in x^9. (4)

4. Below is a system of equations: (4)

$$x + 2y + 3z = 3$$
$$2x - y + 4z = 5$$
$$x - 3y + 2\lambda z = 2.$$

Use Gaussian elimination to find the value of λ which leads to redundancy.

5. Prove by induction that (4)

$$\sum_{r=1}^{n} r(3r - 1) = n^2(n + 1), \ \forall n \in \mathbb{N}.$$

6. (a) Find Maclaurin expansions for $\sin 3x$ and e^{4x} up to and including the term in x^3. (4)

   (b) Hence obtain an expansion for $e^{4x} \sin 3x$ up to and including the term in x^3. (2)

7. **A** is the matrix

$$\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}.$$

   (a) Find the determinant of matrix **A**. (1)

   (b) Show that $\mathbf{A}^2$ can be expressed in the form $p\mathbf{A} + q\mathbf{I}$, stating the values of p and q. (3)

   (c) Obtain a similar expression for $\mathbf{A}^4$. (2)

8. Let $z = \sqrt{3} - i$.

   (a) Plot z on an Argand diagram. (1)

   Let $w = az$, where $a > 0$, $a \in \mathbb{R}$.

   (b) Express w in polar form. (2)

   (c) Express w^8 in the form $ka^n(x + iy)$ where $k, x, y \in \mathbb{Z}$. (3)

9. Obtain (6)

$$\int x^7 (\ln x)^2 \, dx.$$

10. For each of the following statements, decide whether it is true or false.
   If true, give a proof; if false, give a counterexample.

    (a) If a positive integer p is prime, then so is $2p + 1$. (1)

    (b) If a positive integer n has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3. (3)

11. The height of a cube is increasing at the rate of $5 \ \text{cm s}^{-1}$. (4)
    Find the rate of increase of the volume when the height of the cube is 3 cm.

12. Below is a diagram showing the graph of a linear function, $y = f(x)$.
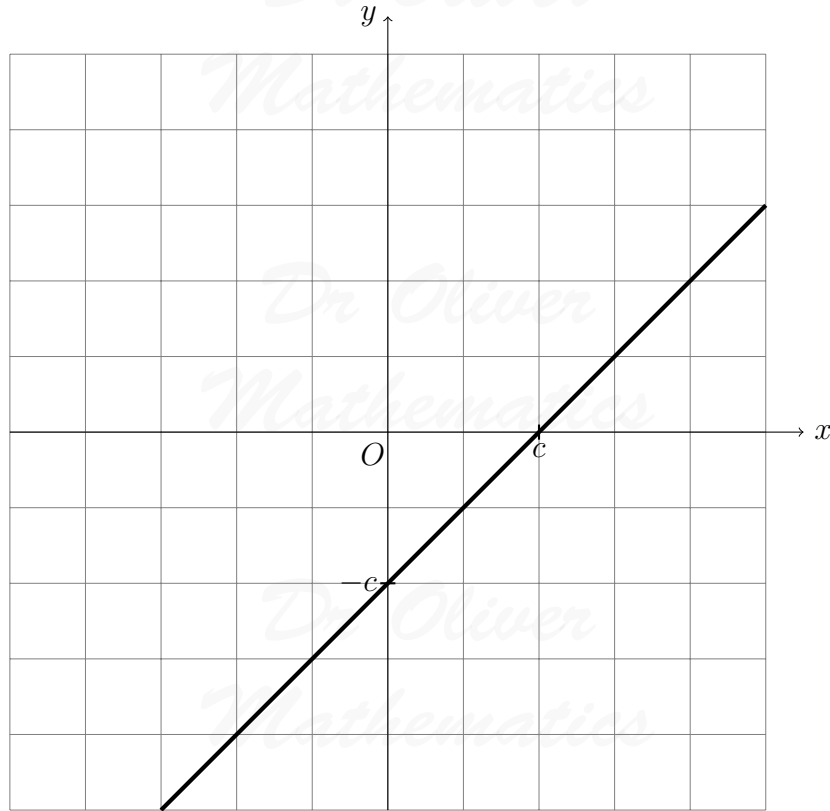
On separate diagrams show:

(a) $y = |\,\mathrm{f}(x) - c\,|,$ (2)

(b) $y = |\,2\,\mathrm{f}(x)\,|.$ (2)

13. (a) Express   (4)
$$\frac{3x + 32}{(x + 4)(6 - x)}$$
in partial fractions.

(b) Hence evaluate   (5)
$$\int_3^4 \frac{3x + 32}{(x + 4)(6 - x)}\,\mathrm{d}x.$$

Give your answer in the form $\ln\left(\dfrac{p}{q}\right).$

14. Two lines L_1 and L_2 are given by the equations:

$$L_1 : \quad x = 4 + 3\lambda,\ y = 2 + 4\lambda,\ z = -7\lambda$$
$$L_2 : \quad \frac{x - 3}{-2} = \frac{y - 8}{1} = \frac{z + 1}{3}.$$

3

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection. (5)

(b) Calculate the obtuse angle between the lines L_1 and L_2. (4)

15. Solve the differential equation (10)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 12x^2 + 2x - 5,$$

given $y = -6$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ when $x = 0$.

16. A beaker of liquid was placed in a fridge. (9)
The rate of cooling is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_F), \; k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The constant temperature in the fridge is $4°C$.
- When first placed in the fridge, the temperature of the liquid was $25°C$.
- At 12 noon, the temperature of the liquid was $9.8°C$.
- At 12:15 pm, the temperature of the liquid had dropped to $6.5°C$.

At what time, to the nearest minute, was the liquid placed in the fridge?

4