Dr Oliver Mathematics
Advance Level Mathematics
Pure Mathematics 1: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Given (3)

$$2^x \times 4^y = \frac{1}{2\sqrt{2}},$$

express y as a function of x.

Solution

$$\begin{aligned} 2^x \times 4^y = \frac{1}{2\sqrt{2}} &\Rightarrow 2^x \times (2^2)^y = \frac{1}{2^{\frac{3}{2}}} \\ &\Rightarrow 2^x \times 2^{2y} = 2^{-\frac{3}{2}} \\ &\Rightarrow 2^{x+2y} = 2^{-\frac{3}{2}} \\ &\Rightarrow x + 2y = -\tfrac{3}{2} \\ &\Rightarrow 2y = -(x + \tfrac{3}{2}) \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{2}(x + \tfrac{3}{2})}}. \end{aligned}$$

2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in ms^{-1}.

Time (s)	0	5	10	15	20	25
Speed (ms^{-1})	2	5	10	18	28	42

Using all of this information,

(a) estimate the length of runway used by the jet to take off. (3)

Solution

$$\begin{aligned}\text{Distance} &\approx \tfrac{1}{2} \times 5 \times [2 + 2(5 + 10 + 18 + 28) + 42] \\ &= \tfrac{1}{2} \times 5 \times 166 \\ &= \underline{\underline{415 \text{ m}}}.\end{aligned}$$

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off. (1)

Solution

Well,

$$\text{area of trapezia} > \text{area under curve}$$

and so it is an overestimate.

3. Figure 1 shows a sector AOB of a circle with centre O, radius 5 cm, and angle $AOB = 40^\circ$.

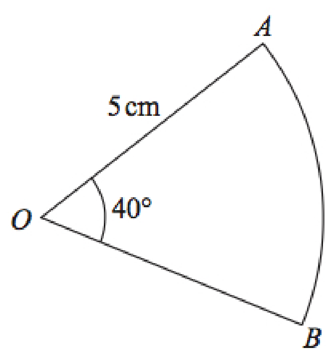

Figure 1: a sector AOB

The attempt of a student to find the area of the sector is shown below.

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 5^2 \times 40$$
$$= 500\,\text{cm}^2$$

(a) Explain the error made by this student. (1)

Solution

Second line: the student should have worked in radians.

(b) Write out a correct solution. (2)

Solution

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times 5^2 \times \left(40 \times \frac{\pi}{180}\right) \\ &= \tfrac{25}{9}\pi.\end{aligned}$$

4. The curve C_1 with parametric equations (6)

$$x = 10\cos t,\ y = 4\sqrt{2}\sin t,\ 0 \leqslant t < 2\pi,$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

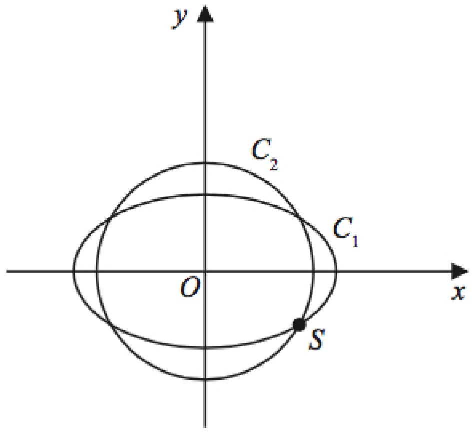

Figure 2: $x = 10\cos t$, $y = 4\sqrt{2}\sin t$ and $x^2 + y^2 = 66$

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S.

Solution

We convert the parametric equations for C^1 into Cartesian coordinates:

$$\begin{aligned}\cos^2 t + \sin^2 t = 1 &\Rightarrow \left(\frac{x}{10}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2 = 1 \\ &\Rightarrow \frac{x^2}{100} + \frac{y^2}{32} = 1 \\ &\Rightarrow x^2 + \frac{25y^2}{8} = 100\end{aligned}$$

and compare the x^2:

$$\begin{aligned}66 - y^2 = 100 - \frac{25y^2}{8} &\Rightarrow \frac{17y^2}{8} = 34 \\ &\Rightarrow y^2 = 16 \\ &\Rightarrow y = \pm 4.\end{aligned}$$

Now, $y = -4$ (why?) and

$$\begin{aligned}x^2 + (-4)^2 = 66 &\Rightarrow x^2 = 50 \\ &\Rightarrow x = \pm 5\sqrt{2};\end{aligned}$$

hence, the Cartesian coordinates of S is $\underline{\underline{(5\sqrt{2}, -4)}}$.

5. Figure 3 shows a sketch of the curve with equation (3)

$$y = \sqrt{x}.$$

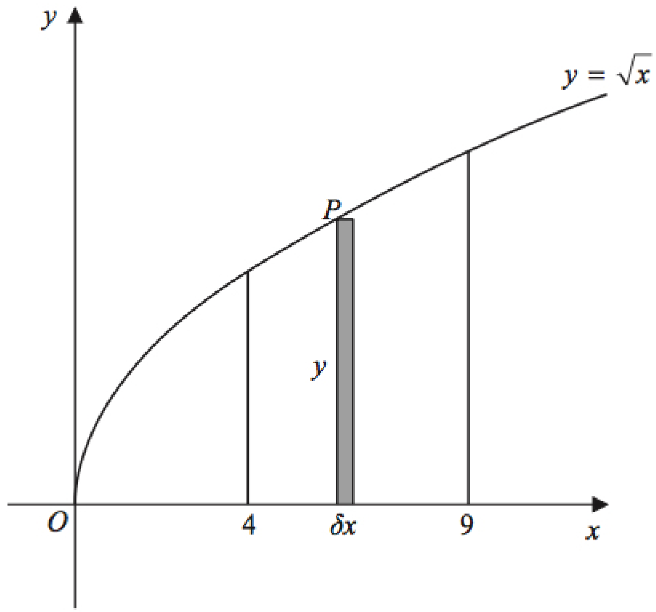

Figure 3: $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.
The rectangle, shown shaded on Figure 3, has height y and width δx.
Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x.$$

Solution

$$\begin{aligned}
\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x &= \int_4^9 \sqrt{x}\, \mathrm{d}x \\
&= \int_4^9 x^{\frac{1}{2}}\, \mathrm{d}x \\
&= \left[\tfrac{2}{3} x^{\frac{3}{2}} \right]_{x=4}^{9} \\
&= \tfrac{2}{3}(27 - 8) \\
&= \underline{\underline{12\tfrac{2}{3}}}.
\end{aligned}$$

6. Figure 4 shows a sketch of the graph of $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \begin{cases} (x-2)^2 + 1 & x \leqslant 2, \\ 4x - 7 & x > 2. \end{cases}$$

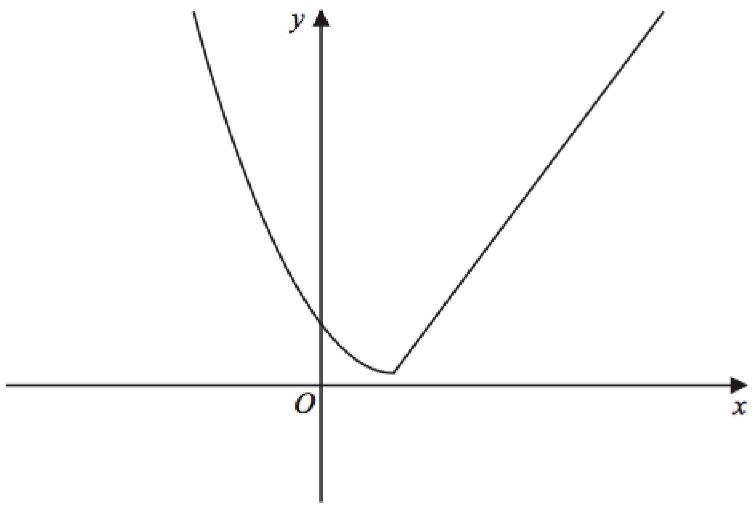

Figure 4: $y = \mathrm{g}(x)$

(a) Find the value of (2)

$$\mathrm{g\,g}(0).$$

Solution

$$\begin{aligned} \mathrm{g\,g}(0) &= \mathrm{g}(\mathrm{g}(0)) \\ &= \mathrm{g}(5) \\ &= \underline{\underline{13}}. \end{aligned}$$

(b) Find all values of x for which (4)

$$\mathrm{g}(x) > 28.$$

Solution

$\underline{x \leqslant 2}$:

$$\begin{aligned}(x-2)^2+1>28 &\Rightarrow (x-2)^2>27\\ &\Rightarrow x-2<-3\sqrt{3}\text{ (yes) or }x-2>3\sqrt{3}\text{ (no)}\\ &\Rightarrow \underline{\underline{x<2-3\sqrt{3}}}.\end{aligned}$$

$\underline{x > 2}$:

$$\begin{aligned}4x-7>28 &\Rightarrow 4x>35\\ &\Rightarrow \underline{\underline{x>8\tfrac{3}{4}}}.\end{aligned}$$

The function h is defined by

$$\mathrm{h}(x)=(x-2)^2+1,\ x\leqslant 2.$$

(c) Explain why h has an inverse but g does not. (1)

Solution

h is a <u>one-to-one function</u> whereas g is a <u>many-to-one function</u>.

(d) Solve the equation (3)

$$\mathrm{h}^{-1}(x)=-\tfrac{1}{2}.$$

Solution

$$\begin{aligned}\mathrm{h}^{-1}(x)=-\tfrac{1}{2} &\Rightarrow x=\mathrm{h}(-\tfrac{1}{2})\\ &\Rightarrow \underline{\underline{x=7\tfrac{1}{4}}}.\end{aligned}$$

7. A small factory makes bars of soap.

On any day, the total cost to the factory, £y, of making x bars of soap is modelled to be the sum of two separate elements:

- a fixed cost,
- a cost that is proportional to the number of bars of soap that are made that day.

(a) Write down a general equation linking y with x, for this model. (1)

Solution

$$\underline{\underline{y = a + bx}}$$

where a and b are fixed constants.

The bars of soap are sold for £2 each.

On a day when 800 bars of soap are made and sold, the factory makes a profit of £500.

On a day when 300 bars of soap are made and sold, the factory makes a loss of £80.

Using the above information,

(b) show that (3)

$$y = 0.84x + 428.$$

Solution

$$\begin{aligned} x = 800 &\Rightarrow y + 500 = 2 \times 800 \\ &\Rightarrow y = 1\,100 \end{aligned}$$

and

$$\begin{aligned} x = 300 &\Rightarrow y - 80 = 2 \times 300 \\ &\Rightarrow y = 680. \end{aligned}$$

So, we have two points: $(800, 1\,100)$ and $(300, 680)$ and we want to know the gradient, b:

$$\begin{aligned} b &= \frac{1\,100 - 680}{800 - 300} \\ &= \frac{420}{500} \\ &= 0.84 \end{aligned}$$

and

$$a = 1\,100 - (0.84 \times 800) = 428.$$

So, we have

$$\underline{\underline{y = 0.84x + 428}},$$

as required.

(c) With reference to the model, interpret the significance of the value 0.84 in the equation. (1)

Solution

They cost of making each extra bar is $\underline{\underline{£0.84}}$.

Assuming that each bar of soap is sold on the day it is made,

(d) find the least number of bars of soap that must be made on any given day for the factory to make a profit that day. (2)

Solution

If the number of bars of soap was n, the number of bars of soap to make a profit would be $2n$:

$$\begin{aligned} 2n \geqslant 0.84n + 428 &\Rightarrow 1.16n \geqslant 428 \\ &\Rightarrow n \geqslant 368\tfrac{28}{29}; \end{aligned}$$

hence, $\underline{\underline{n = 369}}$.

8. (a) Find the value of (3)

$$\sum_{r=4}^{\infty} 20 \times \left(\tfrac{1}{2}\right)^r.$$

Solution

$$\begin{aligned} \sum_{r=4}^{\infty} 20 \times \left(\tfrac{1}{2}\right)^r &= 20 \sum_{r=4}^{\infty} \left(\tfrac{1}{2}\right)^r \\ &= 20 \left[\left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \dots\right] \\ &= 20 \cdot \frac{(\frac{1}{2})^4}{1 - \frac{1}{2}} \\ &= \underline{\underline{2\tfrac{1}{2}}}. \end{aligned}$$

(b) Show that (3)

$$\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1}\right) = 2.$$

Solution

$$\begin{aligned}
& \sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) \\
= \quad & \sum_{n=1}^{48} [\log_5(n+2) - \log_5(n+1)] \\
= \quad & (\log_5 3 - \log_5 2) + (\log_5 4 - \log_5 3) + \ldots + (\log_5 50 - \log_5 48) \\
= \quad & \log_5 50 - \log_5 2 \\
= \quad & \log_5\left(\frac{50}{2}\right) \\
= \quad & \log_5 25 \\
= \quad & \log_5 5^2 \\
= \quad & 2\log_5 5 \\
= \quad & \underline{\underline{2}},
\end{aligned}$$

as required.

9. A research engineer is testing the effectiveness of the braking system of a car when it is driven in wet conditions.

   The engineer measures and records the braking distance, d metres, when the brakes are applied from a speed of V km h^{-1}.

   Graphs of d against V (see Figure 5) and $\log_{10} d$ against $\log_{10} V$ (see Figure 6) were plotted.

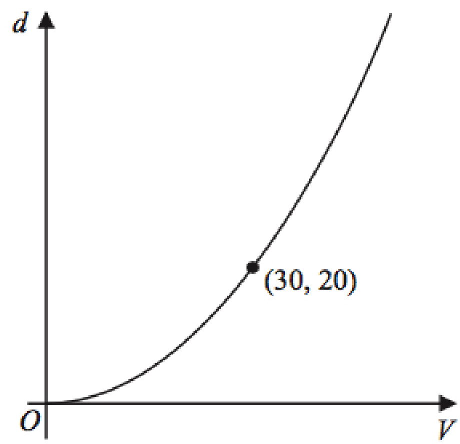

Figure 5: graph of d against V

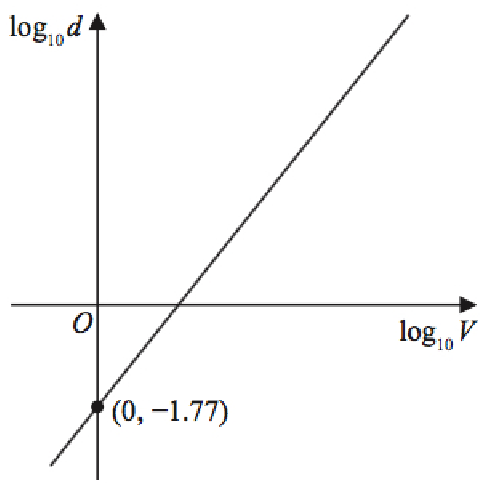

Figure 6: graph of $\log_{10} d$ against $\log_{10} V$

The results are shown below together with a data point from each graph.

(a) Explain how Figure 6 would lead the engineer to believe that the braking distance should be modelled by the formula **(3)**

$$d = kV^n,$$

where k and n are constants with $k \approx 0.017$.

Solution

$$
\begin{aligned}
\log_{10} d = m \log_{10} V + c &\Rightarrow \log_{10} d = \log_{10} V^m + c \\
&\Rightarrow \log_{10} d - \log_{10} V^m = c \\
&\Rightarrow \log_{10}\left(\frac{d}{V^m}\right) = c \\
&\Rightarrow \frac{d}{V^m} = 10^c \\
&\Rightarrow d = 10^c V^m;
\end{aligned}
$$

so, $\underline{\underline{d = kV^n}}$ where $k = 10^c$ and $n = m$.

Now, use $\log_{10} V = 0$ and $\log_{10} d = -1.77$:

$$
\begin{aligned}
\log_{10} d = -1.77 \Rightarrow d &= 10^{-1.77} \\
&= 0.016\,982\,436\,52 \text{ (FCD)} \\
&\approx \underline{\underline{0.017}}.
\end{aligned}
$$

Using the information given in Figure 5, with $k = 0.017$,

(b) find a complete equation for the model giving the value of n to 3 significant figures. (3)

Solution

$$
\begin{aligned}
20 = 0.017(30^n) &\Rightarrow 30^n = \tfrac{20\,000}{17} \\
&\Rightarrow \log 30^n = \log \tfrac{20\,000}{17} \\
&\Rightarrow n \log 30 = \log \tfrac{20\,000}{17} \\
&\Rightarrow n = \frac{\log \tfrac{20\,000}{17}}{\log 30} \\
&\Rightarrow n = 2.078\,760\,335 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{n = 2.08 \text{ (3 sf)}}}.
\end{aligned}
$$

Sean is driving this car at 60 km h^{-1} in wet conditions when he notices a large puddle in the road 100 m ahead. It takes him 0.8 seconds to react before applying the brakes.

(c) Use your formula to find out if Sean will be able to stop before reaching the puddle. (3)

Solution

$$\begin{aligned}\frac{60 \text{ km}}{1 \text{ h}} &\Rightarrow \frac{60\,000 \text{ m}}{60 \text{ min}} \\ &\Rightarrow \frac{60\,000 \text{ m}}{3\,600 \text{ s}} \\ &\Rightarrow 16\tfrac{2}{3} \text{ m s}^{-1}\end{aligned}$$

and he goes

$$16\tfrac{2}{3} \times 0.8 = 13\tfrac{1}{3} \text{ m}$$

in this time. The braking distance is

$$0.017 \cdot (60^{2.08}) = 84.918\,736\,2 \text{ (FCD)}$$

and the overall stopping distance

$$13\tfrac{1}{3} + 84.918\ldots = 98.252\ldots;$$

hence, Sean will be able to stop before reaching the puddle.

10. Figure 7 shows a sketch of triangle OAB.

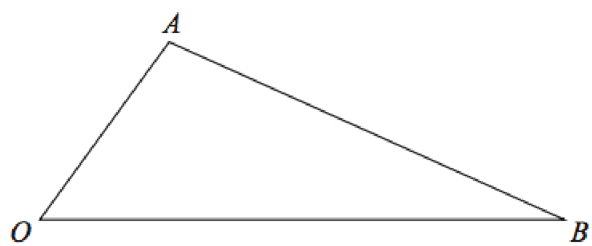

Figure 7: triangle OAB

The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OA}$.
The point M is the midpoint of AB.
The straight line through C and M cuts OB at the point N.
Given $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\overrightarrow{CM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (2)

Solution

$$\begin{aligned}\overrightarrow{CM} &= \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AM} \\ &= -2\mathbf{a} + \mathbf{a} + \tfrac{1}{2}\overrightarrow{AB} \\ &= -\mathbf{a} + \tfrac{1}{2}(-\mathbf{a} + \mathbf{b}) \\ &= \underline{\underline{-\tfrac{3}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}}}.\end{aligned}$$

(b) Show that (2)

$$\overrightarrow{ON} = (2 - \tfrac{3}{2}\lambda)\mathbf{a} + \tfrac{1}{2}\lambda\mathbf{b},$$

where λ is a scalar constant.

Solution

$$\begin{aligned}\overrightarrow{ON} &= \overrightarrow{OC} + \overrightarrow{CN} \\ &= \overrightarrow{OC} + \lambda\overrightarrow{CM} \\ &= 2\mathbf{a} + \lambda(-\tfrac{3}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}) \\ &= \underline{\underline{(2 - \tfrac{3}{2}\lambda)\mathbf{a} + \tfrac{1}{2}\lambda\mathbf{b}}},\end{aligned}$$

as required.

(c) Hence prove that $ON : NB = 2 : 1$. (2)

Solution

Well,

$$\begin{aligned}2 - \tfrac{3}{2}\lambda = 0 &\Rightarrow \tfrac{3}{2}\lambda = 2 \\ &\Rightarrow \lambda = \tfrac{4}{3}\end{aligned}$$

and so

$$\overrightarrow{ON} = \tfrac{2}{3}\mathbf{b}$$

and

$$\overrightarrow{NB} = \tfrac{1}{3}\mathbf{b}.$$

Finally,

$$ON : NB = \tfrac{2}{3}\mathbf{b} : \tfrac{1}{3}\mathbf{b} = \underline{\underline{2 : 1}},$$

as required.

11. Figure 8 shows a sketch of the curve C with equation

$$y = x^x,\ x > 0.$$

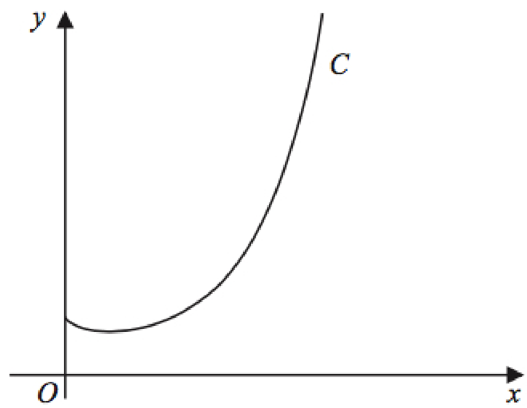

Figure 8: $y = x^x$

(a) Find, by firstly taking logarithms, the x-coordinate of the turning point of C. (5)

Solution

$$\begin{aligned} y = x^x &\Rightarrow \ln y = \ln x^x \\ &\Rightarrow \ln y = x \ln x \\ &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + 1 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x^x(\ln x + 1) \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow x^x(\ln x + 1) = 0 \\ &\Rightarrow \ln x + 1 = 0 \\ &\Rightarrow \ln x = -1 \\ &\Rightarrow \underline{\underline{x = \mathrm{e}^{-1}}}. \end{aligned}$$

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$. (2)

Solution

Let

$$\mathrm{f}(x) = x^x.$$

f(1.5) = 1.837 . . ..
f(1.6) = 2.121 . . ..
The function is continuous and there is a change of sign and so the root lies $\underline{\underline{1.5 < \alpha < 1.6}}$.

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{1-x_n}.$$

Using this formula with $x_1 = 1.5$,

(c) find x_4 to 3 decimal places, (2)

Solution
$x_2 = 1.632\,993\,162$ (FCD)
$x_3 = 1.466\,264\,596$ (FCD)
$x_4 = 1.673\,135\,301$ (FCD) $= \underline{\underline{1.673 \text{ (3 dp)}}}$.

(d) describe the long-term behaviour of x_n. (2)

Solution
It oscillates between 1 and 2.

12. (a) Prove (4)

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta,\ \theta \neq (90n)^\circ,\ n \in \mathbb{Z}.$$

Solution

$$\begin{aligned}\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} &\equiv \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta} \\ &\equiv \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} \\ &\equiv \underline{\underline{2\cot 2\theta}},\end{aligned}$$

as required.

(b) Hence solve, for $90^\circ < \theta < 180^\circ$, the equation (3)

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4,$$

giving any solutions to one decimal place.

Solution

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4 \Rightarrow 2\cot 2\theta = 4$$
$$\Rightarrow \cot 2\theta = 2$$
$$\Rightarrow \tan 2\theta = \tfrac{1}{2}$$
$$\Rightarrow 2\theta = 26.565\,051\,18 \text{ (no)}, 206.565\,051\,18 \text{ (FCD)}$$
$$\Rightarrow \theta = 103.282\,525\,6 \text{ (FCD)}$$
$$\Rightarrow \theta = \underline{\underline{103.3 \text{ (1 dp)}}}.$$

13. A manufacturer produces a storage tank.

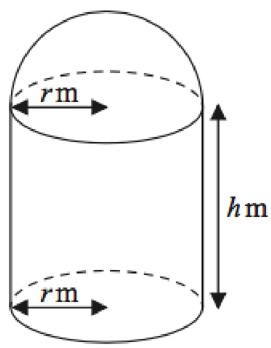

Figure 9: a hollow circular cylinder

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.
The walls of the tank are assumed to have negligible thickness.
The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.
The volume of the tank is 6 m^3.

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by (4)

$$\frac{12}{r} + \tfrac{5}{3}\pi r^2.$$

Solution

The volume is 6:

$$(\pi \cdot r^2 \cdot h) + (\tfrac{2}{3} \cdot \pi \cdot r^3) = 6 \Rightarrow \pi \cdot r^2 \cdot h = 6 - \tfrac{2}{3} \cdot \pi \cdot r^3$$
$$\Rightarrow h = \frac{6 - \frac{2}{3}\pi r^3}{\pi r^2}.$$

Now,

$$\begin{aligned}\text{surface area} &= (2 \cdot \pi \cdot r \cdot h) + (2 \cdot \pi \cdot r^2) + (\pi \cdot r^2)\\ &= 2\pi r\left(\frac{6 - \frac{2}{3}\pi r^3}{\pi r^2}\right) + 3\pi r^2\\ &= \left(\frac{12 - \frac{4}{3}\pi r^3}{r}\right) + 3\pi r^2\\ &= \frac{12}{r} - \tfrac{4}{3}\pi r^2 + 3\pi r^2\\ &= \underline{\underline{\frac{12}{r} + \tfrac{5}{3}\pi r^2}},\end{aligned}$$

as required.

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}r} = 0 &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}(12r^{-1} + \tfrac{5}{3}\pi r^2) = 0\\ &\Rightarrow -12r^{-2} + \tfrac{10}{3}\pi r = 0\\ &\Rightarrow \tfrac{10}{3}\pi r = 12r^{-2}\\ &\Rightarrow r^3 = \frac{36}{10\pi}\\ &\Rightarrow r = 1.046\,447\,736 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{r = 1.05 \text{ m (3 sf)}}}.\end{aligned}$$

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer. (2)

Solution

$$\begin{aligned}\text{Surface area} &= \frac{12}{1.046\ldots} + \tfrac{5}{3}\cdot\pi\cdot 1.046\ldots^2\\ &= 17.201\,050\,016 \text{ (FCD)}\\ &= \underline{\underline{17 \text{ m}^2 \text{ (nearest integer)}}}.\end{aligned}$$

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that (6)

$$\int \frac{1}{4-\sqrt{h}}\,\mathrm{d}h = -8\ln|4-\sqrt{h}| - 2\sqrt{h} + k,$$

where k is a constant.

Solution

$$\begin{aligned}u = 4-\sqrt{h} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}h} &= -\frac{1}{2\sqrt{h}}\\ \Rightarrow \mathrm{d}u &= -\frac{1}{2\sqrt{h}}\,\mathrm{d}h\end{aligned}$$

and

$$\begin{aligned}\int \frac{1}{4-\sqrt{h}}\,\mathrm{d}h &= \int \frac{-2(4-u)}{u}\,\mathrm{d}u\\ &= \int \frac{-2(4-u)}{u}\,\mathrm{d}u\\ &= \int \left(2-\frac{8}{u}\right)\mathrm{d}u\\ &= 2u - 8\ln|u| + k\\ &= -8\ln|4-\sqrt{h}| + 2(4-\sqrt{h}) + c\\ &= \underline{\underline{-8\ln|4-\sqrt{h}| - 2\sqrt{h} + k}},\end{aligned}$$

as required.

A team of scientists is studying a species of slow growing tree.
The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}(4-\sqrt{h})}{20},$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species. (2)

Solution

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = 0 &\Rightarrow \frac{t^{0.25}(4-\sqrt{h})}{20} = 0\\ &\Rightarrow 4-\sqrt{h} = 0\\ &\Rightarrow \sqrt{h} = 4\\ &\Rightarrow h = 16.\end{aligned}$$

Hence, the height is

$$\underline{\underline{0 < h \leqslant 16}}.$$

One of these trees is one metre high when it is first planted.
According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures. (7)

Solution

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}(4-\sqrt{h})}{20} &\Rightarrow \frac{1}{4-\sqrt{h}}\,\mathrm{d}h = \tfrac{1}{20}t^{0.25}\,\mathrm{d}t\\ &\Rightarrow \int \frac{1}{4-\sqrt{h}}\,\mathrm{d}h = \int \tfrac{1}{20}t^{0.25}\,\mathrm{d}t\\ &\Rightarrow -8\ln|4-\sqrt{h}| - 2\sqrt{h} = \tfrac{1}{25}t^{1.25} + c.\end{aligned}$$

Now,

$$t = 0, h = 1 \Rightarrow -8\ln 3 - 2 = c$$

and so

$$-8\ln|4-\sqrt{h}| - 2\sqrt{h} = \tfrac{1}{25}t^{1.25} - 8\ln 3 - 2.$$

Finally,

$$
\begin{aligned}
h = 12 &\Rightarrow -8\ln|4-\sqrt{12}| - 2\sqrt{12} = \tfrac{1}{25}t^{1.25} - 8\ln 3 - 2 \\
&\Rightarrow \tfrac{1}{25}t^{1.25} = -8\ln|4-\sqrt{12}| - 2\sqrt{12} + 8\ln 3 + 2 \\
&\Rightarrow t^{1.25} = -200\ln|4-\sqrt{12}| - 50\sqrt{12} + 200\ln 3 + 50 \\
&\Rightarrow t = 75.154\,151\,68 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{t = 75.2 \text{ years (3 sf)}}}.
\end{aligned}
$$