Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2008 June Paper 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The equation of a curve is given by

$$y = x^2 + ax + 3,$$

where a is a constant.

Given that this equation can also be written as

$$y = (x + 4)^2 + b,$$

find

(a) the value of a and of b, (2)

Solution
Well,

$\times$	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	$+16$

so

$$\begin{aligned} y &= (x + 4)^2 + b \\ &= (x^2 + 8x + 16) + b \\ &= x^2 + 8x + (16 + b). \end{aligned}$$

Hence,

$$\underline{\underline{a = 8}}$$

and

$$16 + b = 3 \Rightarrow \underline{\underline{b = -13}}.$$

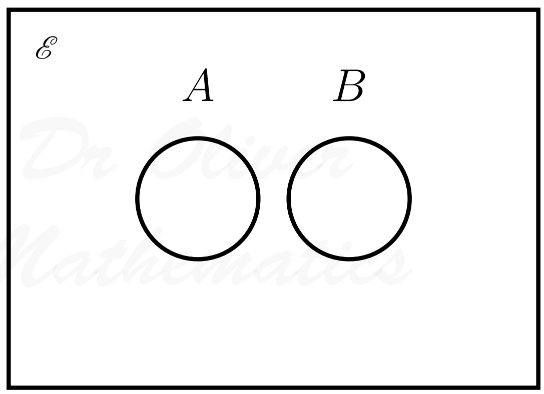
(b) the coordinates of the turning point of the curve. (1)

Solution
The coordinates are $\underline{\underline{(-4, -13)}}$.

2. (a) Illustrate the following statements using a separate Venn diagram for each.

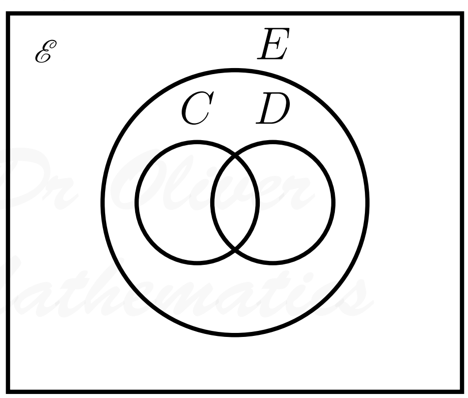
(i) $A \cap B = \varnothing$, (1)

Solution
E.g.,

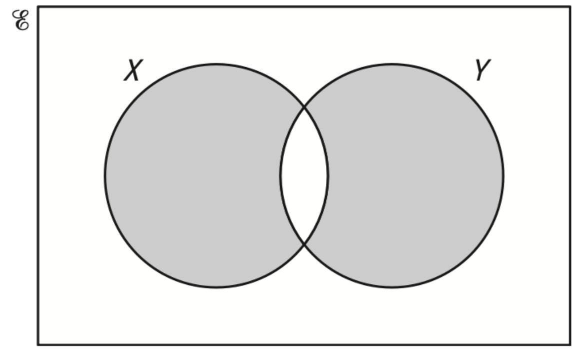


(ii) $(C \cup D) \subset E$. (1)

Solution
E.g.,



(b) Express, in set notation, the set represented by the shaded region. (2)



Solution

E.g., $\underline{\underline{(X \cap Y') \cup (X' \cap Y)}}$.

3. Find the coordinates of the points where the straight line (5)

$$y = 2x - 3$$

intersects the curve

$$x^2 + y^2 + xy + x = 30.$$

Solution

$\times$	$2x$	-3
$2x$	$4x^2$	$-6x$
-3	$-6x$	$+9$

so

$$\begin{aligned} & x^2 + y^2 + xy + x = 30 \\ \Rightarrow\ & x^2 + (2x-3)^2 + x(2x-3) + x = 30 \\ \Rightarrow\ & x^2 + (4x^2 - 12x + 9) + (2x^2 - 3x) + x = 30 \\ \Rightarrow\ & 7x^2 - 14x - 21 = 0 \\ \Rightarrow\ & 7(x^2 - 2x - 3) = 0 \end{aligned}$$

$$\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -3 \end{array} \Bigg\} -3,\ +1$$

$$\begin{aligned} &\Rightarrow \quad 7(x-3)(x+1) = 0 \\ &\Rightarrow \quad x - 3 = 0 \text{ or } x + 1 = 0 \\ &\Rightarrow \quad x = 3 \text{ or } x = -1 \end{aligned}$$

insert the values in to the straight line:

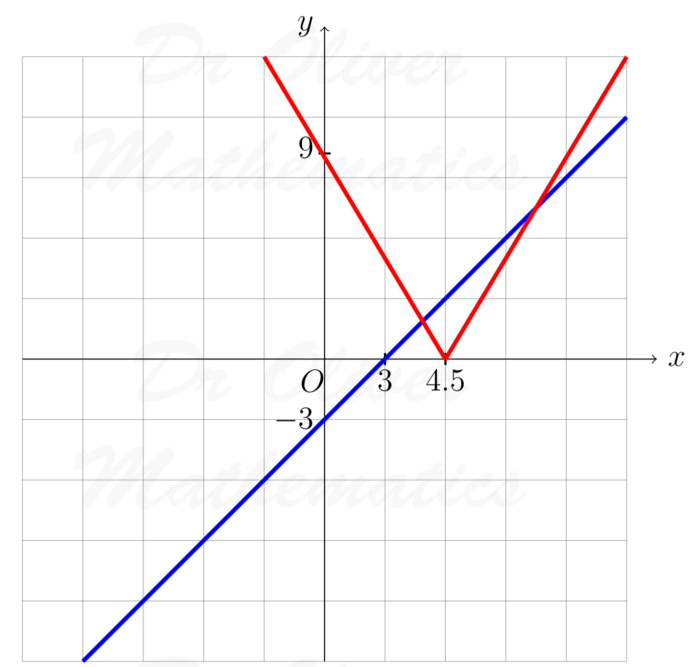
$$\Rightarrow \quad y = 3 \text{ or } y = -5;$$

hence, the coordinates are $\underline{\underline{(3, 3)}}$ and $\underline{\underline{(-1, -5)}}$.

4. (a) Sketch, on the same diagram, the graphs of (3)

$$y = x - 3 \text{ and } y = |2x - 9|.$$

Solution



(b) Solve the equation (2)

$$|2x - 9| = x - 3.$$

Solution
Well,

$$2x - 9 = x - 3 \Rightarrow x = 6$$

and

$$\begin{aligned} -(2x-9) = x - 3 &\Rightarrow -2x + 9 = x - 3 \\ &\Rightarrow -3x = -12 \\ &\Rightarrow x = 4. \end{aligned}$$

The solutions are

$$\underline{\underline{x = 4 \text{ and } x = 6}}.$$

5. Find the coefficient of x^3 in the expansion of

(a) $(1+3x)^8$, (2)

Solution
Well,

$$\begin{aligned} (1+3x)^8 &= \ldots + \binom{8}{2}(1)^6(3x)^2 + \binom{8}{3}(1)^5(3x)^3 + \ldots \\ &= \ldots + 252x^2 + 1\,512x^3 + \ldots; \end{aligned}$$

hence, the coefficient of x^3 is $\underline{\underline{1\,512}}$.

(b) $(1-4x)(1+3x)^8$. (3)

Solution

$\times$	$252x^2$	$+1\,512x^3$
1	$\ldots$	$+1\,512x^3$
$-4x$	$-1\,008x^3$	$\ldots$

Hence, the coefficient of x^3 is

$$1\,512 - 1\,008 = \underline{\underline{504}}.$$

6. (a) Given that (2)

$$\sin x = p,$$

find an expression, in terms of p, for $\sec^2 x$.

Solution

$$\begin{aligned}\sin x = p &\Rightarrow \sin^2 x = p^2 \\ &\Rightarrow 1 - \cos^2 x = p^2 \\ &\Rightarrow -\cos^2 x = -1 + p^2 \\ &\Rightarrow \cos^2 x = 1 - p^2 \\ &\Rightarrow \underline{\underline{\sec^2 x = \frac{1}{1-p^2}}}.\end{aligned}$$

(b) Prove that (4)

$$\sec A \operatorname{cosec} A - \cot A \equiv \tan A.$$

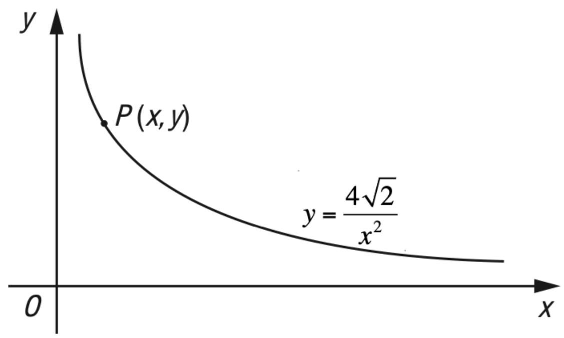
Solution

$$\begin{aligned}\sec A \operatorname{cosec} A - \cot A &\equiv \left(\frac{1}{\cos A}\right)\left(\frac{1}{\sin A}\right) - \frac{\cos A}{\sin A} \\ &\equiv \frac{1}{\sin A \cos A} - \frac{\cos A}{\sin A} \\ &\equiv \frac{1}{\sin A \cos A} - \frac{\cos^2 A}{\sin A \cos A} \\ &\equiv \frac{1 - \cos^2 A}{\sin A \cos A} \\ &\equiv \frac{\sin^2 A}{\sin A \cos A} \\ &\equiv \frac{\sin A}{\cos A} \\ &\equiv \underline{\underline{\tan A}},\end{aligned}$$

as required.

7. The diagram shows part of the curve

$$y = \frac{4\sqrt{2}}{x^2}.$$



The point $P(x, y)$ lies on this curve.

(a) Write down an expression, in terms of x, for $(OP)^2$. (1)

Solution

$$(OP)^2 = x^2 + \left(\frac{4\sqrt{2}}{x^2}\right)^2$$
$$= \underline{\underline{x^2 + \frac{32}{x^4}}}.$$

(b) Denoting $(OP)^2$ by S, find an expression for $\frac{\mathrm{d}S}{\mathrm{d}x}$. (2)

Solution

$$S = x^2 + \frac{32}{x^4} \Rightarrow S = x^2 + 32x^{-4}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}S}{\mathrm{d}x} = 2x - 128x^{-5}}}.$$

(c) Find the value of x for which S has a stationary value and the corresponding value of OP. (3)

Solution

$$\frac{\mathrm{d}S}{\mathrm{d}x} = 0 \Rightarrow 2x - 128x^{-5} = 0$$
$$\Rightarrow 2x = \frac{128}{x^5}$$
$$\Rightarrow x = \frac{64}{x^5}$$
$$\Rightarrow x^6 = 64$$
$$\Rightarrow x = \pm 2;$$

but $x = -2$ is not on the curve so $x = 2$. Hence,

$$(OP)^2 = 2^2 + \frac{32}{2^4} \Rightarrow (OP)^2 = 6$$
$$\Rightarrow \underline{\underline{OP = \sqrt{6}}}.$$

8. Solve the equation

(a) $2^{2x+1} = 20$, (3)

Solution

$$2^{2x+1} = 20 \Rightarrow \log_2 2^{2x+1} = \log_2 20$$
$$\Rightarrow 2x + 1 = \log_2 20$$
$$\Rightarrow 2x = \log_2 20 - 1$$
$$\Rightarrow \underline{\underline{x = \tfrac{1}{2}(\log_2 20 - 1) \text{ or } 1.66 \text{ (3 sf)}}}.$$

(b) (4)

$$\frac{5^{4y-1}}{25^y} = \frac{125^{y+3}}{25^{2-y}}.$$

Solution

$$\frac{5^{4y-1}}{25^y} = \frac{125^{y+3}}{25^{2-y}} \Rightarrow \frac{5^{4y-1}}{(5^2)^y} = \frac{(5^3)^{y+3}}{(5^2)^{2-y}}$$

$$\Rightarrow \frac{5^{4y-1}}{5^{2y}} = \frac{5^{3(y+3)}}{5^{2(2-y)}}$$

$$\Rightarrow 5^{(4y-1)-2y} = 5^{3(y+3)-2(2-y)}$$

$$\Rightarrow 5^{2y-1} = 5^{5y+13}$$

take $\log_5$:

$$\Rightarrow 2y - 1 = 5y + 5$$

$$\Rightarrow -6 = 3y$$

$$\Rightarrow \underline{\underline{y = -2}}.$$

9. Given that

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 4 \\ 1 \end{pmatrix},$$

calculate

(a) **AB**, (2)

Solution

$$\mathbf{AB} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 12 & -18 \\ 6 & -4 \end{pmatrix}}}.$$

(b) **BC**, (2)

Solution

$$\mathbf{BC} = \begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 7 \\ 2 \end{pmatrix}}}.$$

(c) the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. (4)

Solution

$$\begin{aligned}\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \Rightarrow \mathbf{X} &= \tfrac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix} \\ \Rightarrow \mathbf{X} &= \tfrac{1}{10}\begin{pmatrix} 9 & -17 \\ -6 & 18 \end{pmatrix} \\ \Rightarrow \mathbf{X} &= \begin{pmatrix} 0.9 & -1.7 \\ -0.6 & 1.8 \end{pmatrix}.\end{aligned}$$

10. Find

(a) (i) (2)

$$\int \frac{12}{(2x-1)^4}\,\mathrm{d}x,$$

Solution

$$\begin{aligned}\int \frac{12}{(2x-1)^4}\,\mathrm{d}x &= 12\int (2x-1)^{-4}\,\mathrm{d}x \\ &= 12\left[-\tfrac{1}{6}(2x-1)^{-3}\right] + c \\ &= -2(2x-1)^{-3} + c.\end{aligned}$$

(ii) (3)

$$\int x(x-1)^2\,\mathrm{d}x.$$

Solution

$\times$	x	-1
x	x^2	$-x$
-1	$-x$	$+1$

so

$$\begin{aligned}\int x(x-1)^2\,\mathrm{d}x &= \int x(x^2-2x+1)\,\mathrm{d}x \\ &= \int (x^3-2x^2+x)\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2 + c}}.\end{aligned}$$

(b) (i) Given that (3)

$$y = 2(x-5)\sqrt{x+4},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(x+1)}{\sqrt{x+4}}.$$

Solution

Product rule:

$$\begin{aligned}u = 2(x-5) &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \\ v = (x+4)^{\frac{1}{2}} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \tfrac{1}{2}(x+4)^{-\frac{1}{2}}\end{aligned}$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= [2(x-5)][\tfrac{1}{2}(x+4)^{-\frac{1}{2}}] + [2][(x+4)^{\frac{1}{2}}] \\ &= \frac{x-5}{(x+4)^{\frac{1}{2}}} + 2(x+4)^{\frac{1}{2}} \\ &= \frac{x-5}{(x+4)^{\frac{1}{2}}} + \frac{2(x+4)}{(x+4)^{\frac{1}{2}}} \\ &= \frac{(x-5)+2(x+4)}{\sqrt{x+4}} \\ &= \frac{x-5+2x+8}{\sqrt{x+4}} \\ &= \frac{3x+3}{\sqrt{x+4}} \\ &= \underline{\underline{\frac{3(x+1)}{\sqrt{x+4}}}},\end{aligned}$$

as required.

(ii) Hence find (2)

$$\int \frac{x+1}{\sqrt{x+4}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{x+1}{\sqrt{x+4}}\,\mathrm{d}x &= \tfrac{1}{3}\int \frac{3(x+1)}{\sqrt{x+4}}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{2}{3}(x-5)\sqrt{x+4}+c}}.\end{aligned}$$

11. The function f is defined by

$$\mathrm{f}(x) = (x+1)^2 + 2, \text{ for } x \geqslant -1.$$

Find

(a) the range of f, (1)

Solution

$\underline{\underline{\mathrm{f}(x) \geqslant 2}}$.

(b) $\mathrm{f}^2(1)$, (1)

Solution

$$\begin{aligned}\mathrm{f}^2(1) &= \mathrm{f}(\mathrm{f}(1)) \\ &= \mathrm{f}(6) \\ &= \underline{\underline{51}}.\end{aligned}$$

(c) an expression for $\mathrm{f}^{-1}(x)$. (3)

Solution

$$\begin{aligned}y = (x+1)^2 + 2 &\Rightarrow y - 2 = (x+1)^2 \\ &\Rightarrow \pm\sqrt{y-2} = x+1 \\ &\Rightarrow \pm\sqrt{y-2} - 1 = x.\end{aligned}$$

Which is it: plus or minus? Hence,

$$\underline{\underline{y = \sqrt{x-2} - 1, \text{ for } x \geqslant 2}}.$$

The function g is defined by

$$\mathrm{g}(x) = \frac{20}{x+1}, \text{ for } x \geqslant 0.$$

Find

(d) $\mathrm{g}^{-1}(2)$, (2)

Solution

$$\begin{aligned} y = \frac{20}{x+1} &\Rightarrow x + 1 = \frac{20}{y} \\ &\Rightarrow x = \frac{20}{y} - 1 \end{aligned}$$

and so

$$\mathrm{g}^{-1}(x) = \frac{20}{x} - 1.$$

Finally,

$$\mathrm{g}^{-1}(2) = \frac{20}{2} - 1 = \underline{\underline{9}}.$$

(e) the value of x for which (4)

$$\mathrm{f\,g}(x) = 38.$$

Solution

$$\begin{aligned} \mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}\left(\frac{20}{x+1}\right) \\ &= \left(\frac{20}{x+1} + 1\right)^2 + 2. \end{aligned}$$

Now,

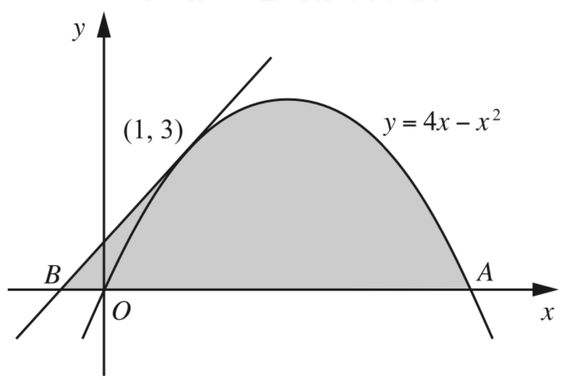
$$\begin{aligned} \mathrm{f}\,\mathrm{g}(x) = 38 &\Rightarrow \left(\frac{20}{x+1}+1\right)^2 + 2 = 38 \\ &\Rightarrow \left(\frac{20}{x+1}+1\right)^2 = 36 \\ &\Rightarrow \frac{20}{x+1}+1 = 6 \\ &\Rightarrow \frac{20}{x+1} = 5 \\ &\Rightarrow \frac{1}{x+1} = \tfrac{1}{4} \\ &\Rightarrow x+1 = 4 \\ &\Rightarrow \underline{\underline{x = 3}}. \end{aligned}$$

EITHER

12. The diagram shows the curve

$$y = 4x - x^2,$$

which crosses the x-axis at the origin O and the point A.



The tangent to the curve at the point $(1, 3)$ crosses the x-axis at the point B.

(a) Find the coordinates of A and of B. (5)

Solution

For A,

$$4x - x^2 = 0 \Rightarrow x(4-x) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 4.$$

So, $\underline{\underline{A(4,0)}}$.

For B,

$$y = 4x - x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4 - 2x$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

Now, the equation of the tangent is

$$y - 3 = 2(x-1)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -3 = 2(x-1) \\ &\Rightarrow -\tfrac{3}{2} = x - 1 \\ &\Rightarrow x = -\tfrac{1}{2}. \end{aligned}$$

So, $\underline{\underline{B(-\frac{1}{2}, 0)}}$.

(b) Find the area of the shaded region. (5)

Solution

$$\begin{aligned} \text{Area} &= \text{area under } [-\tfrac{1}{2}, 1] + \text{area under } [1, 4] \\ &= (\tfrac{1}{2} \times \tfrac{3}{2} \times 3) + \int_1^4 (4x - x^2)\,\mathrm{d}x \\ &= 2\tfrac{1}{4} + \left[2x^2 - \tfrac{1}{3}x^3\right]_{x=1}^4 \\ &= 2\tfrac{1}{4} + \left(32 - 21\tfrac{1}{3}\right) - \left(2 - \tfrac{1}{3}\right) \\ &= 2\tfrac{1}{4} + 9 \\ &= \underline{\underline{11\tfrac{1}{4}}}. \end{aligned}$$

OR

13. The points $A(-2, 2)$, $B(4, 4)$, and $C(5, 2)$ are the vertices of a triangle. (10)

The perpendicular bisector of AB and the line through A parallel to BC intersect at the point D.

Find the area of the quadrilateral $ABCD$.

Solution

The perpendicular bisector of AB:

Well,

$$\left(\frac{-2+4}{2}, \frac{2+4}{2}\right) = (1, 3)$$

and

$$\begin{aligned} m &= \frac{4-2}{4-(-2)} \\ &= \tfrac{1}{3} \end{aligned}$$

and

$$m_{\text{normal}} = -3.$$

The perpendicular bisector of AB is

$$\begin{aligned} y - 3 = -3(x-1) &\Rightarrow y - 3 = -3x + 3 \\ &\Rightarrow \boxed{y = -3x + 6 \quad (1).} \end{aligned}$$

The line through A parallel to BC:

$$\begin{aligned} m &= \frac{2-4}{5-4} \\ &= -2 \end{aligned}$$

and the line through A parallel to BC is

$$\begin{aligned} y - 2 = -2(x+2) &\Rightarrow y - 2 = -2x - 4 \\ &\Rightarrow \boxed{y = -2x - 2 \quad (2).} \end{aligned}$$

Do (1) − (2):

$$\begin{aligned} 0 = -x + 8 &\Rightarrow x = 8 \\ &\Rightarrow y = -18; \end{aligned}$$

hence, $D(8, -18)$.

As AC is horizontal,

$$\begin{aligned}\text{area of the quadrilateral} &= \text{area of } ABC + \text{area of } ACD \\ &= (\tfrac{1}{2} \times 7 \times 2) + (\tfrac{1}{2} \times 7 \times 20) \\ &= 7 + 70 \\ &= \underline{\underline{77}}.\end{aligned}$$