Dr Oliver Mathematics
Lower and Upper Bounds

Suppose that we are told that the value of a number x is 2.76, correct to 2 decimal places. What possible values can x take?

x must, in fact, satisfy the inequalities

$$2.755 \leqslant x < 2.765,$$

and we need strict inequality on the right since 2.765, if given to 2 decimal places, is 2.77.

The number on the left of the inequality statement above is the *lower bound* (and it is attainable) for x and the number on the right is the upper bound (and it is not attainable) for x.

x *could* be 2.7649; x *could* be 2.764 999 99; but x *cannot* be $2.764\dot{9}$ since this number is exactly the same as 2.765.

Let L_a and U_a be the lower and upper bounds of a respectively and let L_b and U_b be the lower and upper bounds of b respectively. Then

(a) $L_a + L_b \leqslant a + b < U_a + U_b$,

(b) $L_a - U_b < a - b < U_a + L_b$,

(c) $L_a L_b \leqslant ab < U_a U_b$,

(d) $\dfrac{L_a}{U_b} < \dfrac{a}{b} < \dfrac{U_a}{L_b}$.

Note that, for sums and products, we have an attainable lower bound but an unattainable upper bound and that, for differences and quotients, we have unattainable lower and upper bounds.

We look at a few examples.

1. Martin won the 400 metre race in the school sports with a time of 1 minute.
   The distance was correct to the nearest centimetre.
   The time was correct to the nearest tenth of a second.

   (a) Work out the upper bound and the lower bound of Martin's speed in km/h. (5)
   Give your answers correct to 5 significant figures.

Solution

We will start by woking out the race distance. The question states "The distance was correct to the nearest centimetre" which means

$$\text{least distance} = 400 - 0.005 = 399.995$$

and

$$\text{upper distance} = 400 + 0.005 = 400.005.$$

So, we have for the race distance,

$$399.995 \leqslant \text{race distance} < 400.005.$$

The question states "The time was correct to the nearest tenth of a second" which means

$$\text{least time} = 60 - 0.05 = 59.95$$

and

$$\text{upper time} = 60 + 0.05 = 60.05.$$

So, we have for the time:

$$59.95 \leqslant \text{time} < 60.05.$$

And we have

$$\frac{399.995}{60.05} < \text{speed (m/s)} < \frac{400.005}{59.95}.$$

But we want km/h and so we multiply by 60×60 and divide by 1000:

$$\frac{399.995 \times 60 \times 60}{60.05 \times 1000} < \text{speed (km/h)} < \frac{400.005 \times 60 \times 60}{59.95 \times 1000}.$$

Finally,

$$\text{lower bound} = 23.979\,716\,9 \text{ (FCD)} = \underline{\underline{23.980 \text{ km/h (5 sf)}}}$$

and

$$\text{upper bound} = 24.020\,316\,93 \text{ (FCD)} = \underline{\underline{24.020 \text{ km/h (5 sf)}}}.$$

(b) Write down an appropriate value for Martin's speed in km/h. (1)
Explain your answer.

Solution

We draw up a table.

	Lower bound	Upper bound	Agree?
1 sf	20	20	Yes
2 sf	24	24	Yes
3 sf	24.0	24.0	Yes
4 sf	23.98	24.02	No

So, an appropriate value for Martin's speed is 24.0 km/h (3 sf).

2. The length of a rectangle is 6.7 cm, correct to 2 significant figures.

   (a) For the length of the rectangle, write down (2)

   (i) the upper bound,

   Solution

   6.75 cm.

   (ii) the lower bound

   Solution

   6.65 cm.

   The area of the rectangle is 26.9 cm^2, correct to 3 significant figures.

   (b) (i) Calculate the upper bound for the width of the rectangle. Write down all the figures on your calculator display. (3)

   Solution

   The area of the rectangle is

$$26.85 \leqslant \text{area} < 26.95$$

   and so the width equals

$$\text{upper bound} = \frac{26.95}{6.65} = 4\tfrac{1}{19} = 4.052\,631\,579 \text{ cm (FCD)}.$$

(ii) Calculate the lower bound for the width of the rectangle.
Write down all the figures on your calculator display.

Solution

$$\text{Lower bound} = \frac{26.85}{6.75} = 3\tfrac{44}{45} = \underline{\underline{3.977\,777\,778 \text{ cm (FCD)}}}.$$

(c) (i) Write down the width of the rectangle to an appropriate degree of accuracy. (2)

Solution

$\underline{\underline{4}}$.

(ii) Give a reason for your answer.

Solution

	Lower bound	Upper bound	Agree?
1 sf	4	4	Yes
2 sf	4.0	4.1	No

So, an appropriate value the width of the rectangle is $\underline{\underline{4 \text{ cm (1 sf)}}}$.

3. Kelly runs a distance of 100 metres in a time of 10.52 seconds.
The distance of 100 metres was measured to the nearest metre.
The time of 10.52 seconds was measured to the nearest hundredth of a second.

(a) Write down the upper bound for the distance of 100 metres. (1)

Solution

$$99.5 \leqslant \text{distance} < 100.5$$

and

$$\text{distance} = \underline{\underline{100.5 \text{ m}}}.$$

(b) Write down the lower bound for the time of 10.52 seconds. (1)

Solution

$$10.515 \leqslant \text{time} < 10.525$$

and

$$\text{time} = \underline{\underline{10.515 \text{ s}}}.$$

(c) Calculate the upper bound for Kelly's average speed. (2)
Write down all the figures on your calculator display.

Solution

$$\text{Upper bound} = \frac{100.5}{10.515} = \underline{\underline{9.557\,774\,608 \text{ m/s (FCD)}}}.$$

(d) Calculate the loweer bound for Kelly's average speed. (2)
Write down all the figures on your calculator display.

Solution

$$\text{Lower bound} = \frac{99.5}{10.525} = \underline{\underline{9.453\,681\,71 \text{ m/s (FCD)}}}.$$

4. Steve travelled from Ashton to Barnfield. (4)

He travelled 235 miles, correct to the nearest 5 miles.
The journey took him 200 minutes, correct to the nearest 5 minutes.

Calculate the lower bound for the average speed of the journey.
Give your answer in miles per hour, correct to 3 significant figures.
You must show all your working.

Solution

$$232.5 \leqslant \text{length (km)} < 237.5$$

and

$$197.5 \leqslant \text{minutes (min)} < 202.5.$$

But we want km/h and so

$$\begin{aligned}\text{lower bound} &= \frac{232.5 \times 60}{202.5} \\ &= 68.888\,888\,889 \text{ (FCD)} \\ &= \underline{\underline{68.9 \text{ km/h (FCD)}}}.\end{aligned}$$

5. A train travelled along a track in 110 minutes, correct to the nearest 5 minutes.
Jake finds out that the track is 270 km long.
He assumes that the track has been measured correct to the nearest 10 km.

(a) Could the average speed of the train have been greater than 160 km/h? You must show how you get your answer. (4)

Solution

$$265 \leqslant \text{length (km)} < 275$$

and

$$107.5 \leqslant \text{minutes (min)} < 112.5.$$

But we want km/h and so

$$\text{the average speed} = \frac{275 \times 60}{107.5} = 153\tfrac{21}{43}$$

and the answer is $\underline{\underline{\text{no}}}$.

Jake's assumption was wrong.
The track was measured correct to the nearest 5 km.

(b) Explain how this could affect your decision in part (a). (1)

Solution

This time,

$$267.5 \leqslant \text{length (km)} < 272.5$$

and $\underline{\underline{\text{the average speed would drop}}}$.

6. A shoebox is in the shape of a cuboid and has a volume of 8300 cm^3, correct to 2 significant figures. (4)

The length of the box is 33 cm correct to the nearest cm.
The width of the box is 21 cm correct to the nearest cm.

Calculate the upper bound for the height of the box.
Write down all the figures on your calculator display.

Solution

$$8250 \leqslant \text{volume} < 8350,$$

$$32.5 \leqslant \text{length} < 33.5,$$

and

$$20.5 \leqslant \text{width} < 21.5.$$

Finally,

$$\begin{aligned}\text{height} &= \frac{8350}{32.5 \times 20.5} \\ &= 12\tfrac{284}{533} \\ &= \underline{\underline{12.532\,833\,02 \text{ cm (FCD)}}}.\end{aligned}$$