Dr Oliver Mathematics
Advance Level Further Mathematics
Further Mathematics 3: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that, for $x \in \mathbb{R}$, **(2)**

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

   (b) Hence, given that $-1 < \theta < 1$, prove that **(3)**

$$\operatorname{artanh} \theta = \tfrac{1}{2} \ln \left(\frac{1 + \theta}{1 - \theta} \right).$$

2. Figure 1 shows a sketch of part of the curve with equation

$$y = 5 \cosh x - 6 \sinh x.$$

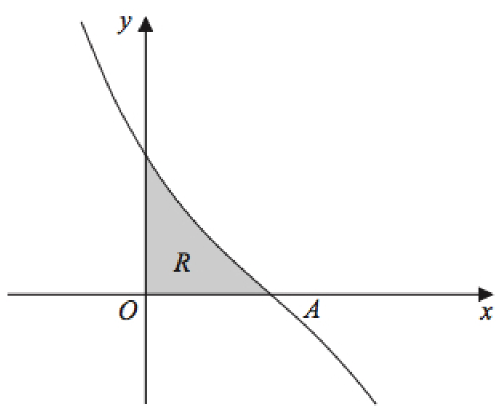

Figure 1: $y = 5 \cosh x - 6 \sinh x$

The curve crosses the x-axis at the point A.

   (a) Find the exact value of the x-coordinate of the point A, giving your answer as a natural logarithm. **(3)**

(b) Show that (3)

$$(5\cosh x - 6\sinh x)^2 \equiv a\cosh 2x + b\sinh 2x + c,$$

where a, b, and c are constants to be found.

The finite region R, bounded by the curve and the coordinate axes, is shown shaded in Figure 1.

The region R is rotated through 2π radians about the x-axis.

(c) Use calculus to find the volume of the solid generated, giving your answer as an exact multiple of π. (4)

3.

$$\mathbf{M} = \begin{pmatrix} 3 & k & 2 \\ -1 & 0 & 1 \\ 1 & k & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that 3 is an eigenvalue of $\mathbf{M}$,

(a) find the value of k. (3)

(b) Hence find the other two eigenvalues of $\mathbf{M}$. (4)

(c) Find an eigenvector corresponding to the eigenvalue 3. (2)

4. The curve C has equation

$$y = \operatorname{arsinh} x + x\sqrt{x^2+1},\ 0 \leqslant x \leqslant 1.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{x^2+1}.$$

(b) Hence show that the length of the curve C is given by (2)

$$\int_0^1 \sqrt{4x^2+5}\,\mathrm{d}x.$$

(c) Using the substitution (6)

$$x = \tfrac{\sqrt{5}}{2}\sinh u,$$

find the exact length of the curve C, giving your answer in the form $a + b\ln c$, where a, b, and c are constants to be found.

5. Given that

$$I_n = \int x^n\sqrt{x+8}\,\mathrm{d}x,\ n \geqslant 0,\ x \geqslant 0,$$

(a) show that, for $n \geqslant 1$, (6)

$$I_n = \frac{px^n(x+8)^{\frac{3}{2}}}{2n+3} - \frac{qn}{2n+3}I_{n-1},$$

where p and q are constants to be found.

(b) Use part (a) to find the exact value of (5)

$$\int_0^{10} x^2\sqrt{x+8}\,\mathrm{d}x,$$

giving your answer in the form $k\sqrt{2}$, where k is rational.

6. The line l_1 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

where λ is a scalar parameter.

The line l_2 has equation

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-1}{3}.$$

(a) Prove that the lines l_1 and l_2 are skew. (4)

(b) Find the shortest distance between the lines l_1 and l_2. (5)

The plane Π contains l_1 and intersects l_2 at the point $(3, 8, 13)$.

(c) Find a cartesian equation for the plane Π. (4)

7. The ellipse E has foci at the points $(\pm 3, 0)$ and has directrices with equations $x = \pm\frac{25}{3}$.

(a) Find a cartesian equation for the ellipse E. (5)

The straight line l has equation $y = mx + c$, where m and c are **positive** constants.

(b) Show that the x-coordinates of any points of intersection of l and E satisfy the equation (2)

$$(16 + 25m^2)x^2 + 50mcx + 25(c^2 - 16) = 0.$$

Given that the line l is a tangent to E,

(c) show that (3)

$$c^2 = pm^2 + q,$$

where p and q are constants to be found.

The line l intersects the x-axis at the point A and intersects the y-axis at the point B.

(d) Show that the area of triangle OAB, where O is the origin, is (3)

$$\frac{25m^2 + 16}{2m}.$$

(e) Find the minimum area of triangle OAB. (2)